Exercise 1. Energy preservation

Suppose that the system is characterized by a Hamiltonian H, and a unitary operation U is applied.

(a) Show that if $[U, H] = 0$, then the unitary preserves the energy of the system.

(b) Consider a four-level system with a Hamiltonian $H = \Delta|1\rangle\langle 1| + \Delta|2\rangle\langle 2| + 2\Delta|3\rangle\langle 3|$, written in the energy eigenbasis $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. Come up with one non-trivial unitary U_{pres} which would preserve the energy of the system for any state, and identify the common eigenbasis of U and H. Find another unitary $U_{\text{non-pres}}$ which would not preserve the energy of the system for any state.

(c) Give an example of an initial state of the system, for which the energy would still be preserved after applying $U_{\text{non-pres}}$.

Exercise 2. Temperature of a qubit

Consider a qubit with Hamiltonian $H = \Delta|1\rangle\langle 1|$, with the energy gap $\Delta = 1$ for simplicity. Recall that the thermal state at temperature T is given by

$$\tau(T) = \frac{e^{-\frac{H}{kT}}}{Z},$$

where k is a constant (Boltzmann constant), and Z is the normalization factor, which is called the *partition function*:

$$Z(T, H) = \sum_i e^{-\frac{E_i}{kT}}.$$

For simplicity of notation, we can define the *inverse temperature* $\beta = \frac{1}{kT}$.

(a) What is the temperature of the state $\rho = |0\rangle\langle 0|$ (the ground state)? And of $\rho = |1\rangle\langle 1|$ (the excited state)? And of the maximally mixed state $\rho = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$? Place these three states on the β axis.

(b) Take the following two states that are close to the maximally mixed state, $\epsilon > 0$:

$$\rho_+ = \left(\frac{1}{2} + \epsilon\right)|0\rangle\langle 0| + \left(\frac{1}{2} - \epsilon\right)|1\rangle\langle 1| \tag{1}$$

$$\rho_- = \left(\frac{1}{2} - \epsilon\right)|0\rangle\langle 0| + \left(\frac{1}{2} + \epsilon\right)|1\rangle\langle 1|. \tag{2}$$

Find the temperatures $\beta_\pm$ corresponding to the above states, using $\epsilon << 1$ to expand to first order approximations.

(c) Continuing from the above, look at the limit $\epsilon \to 0$ to argue that β is continuous w.r.t. the populations, while $T = (k\beta)^{-1}$ is not, which is why it is more natural to work with β in quantum thermodynamics. (Another way to argue this is to show that the limit $T \to 0$ is not the same from the left and right, by showing that the states in either limit are very different).