Exercise 1. Cooling a real qubit with a virtual one

In this exercise, we consider the process of cooling the qubit by repeatedly swapping it with a resetted virtual qubit. Let S be a real qubit which is initially in the fully mixed state:

$$\rho_S = \frac{1}{2}|0\rangle\langle 0|_S + \frac{1}{2}|1\rangle\langle 1|_S.$$

Our goal is to cool this qubit to its ground state $|0\rangle\langle 0|_S$. For this we can use a larger dimensional system V, and choose a virtual qubit with temperature $\beta_V = +\infty$ (all population is concentrated on the level with lower energy) and norm N_V.

(a) What is the reduced state of the qubit S after applying the swapping map n times? The virtual qubit is reset at each iteration.

(b) How many times do you need to apply the swap to get a state of S, which is ϵ-close to the ground state $\rho_S^{(N)} = (1-\epsilon)|0\rangle\langle 0|_S + \epsilon|1\rangle\langle 1|_S$, $\epsilon \ll 1$?

(c) Assuming that the energy gaps of the real and virtual qubits are E_S and E_V respectively, calculate the change in the energy of the system after each step. What is the total energy cost of cooling to $|0\rangle\langle 0|_S$?

Exercise 2. Optimize the cooling of qubit with a qutrit

Let us take a qutrit B as an example of a multi-dimensional system to take virtual qubits from, described by a Hamiltonian

$$H_B = E_B|1\rangle\langle 1|_B + 2E_B|2\rangle\langle 2|_B.$$

The qutrit is initially in the state

$$\rho_B = \frac{2}{3}|0\rangle\langle 0|_B + \frac{1}{4}|1\rangle\langle 1|_B + \frac{1}{12}|2\rangle\langle 2|_B.$$

Let us also consider a qubit S, initially in the fully mixed state.

(a) Characterize two virtual qubits of V: one making use of the eigenstates $|0\rangle$ and $|1\rangle$, and one – of the eigenstates $|1\rangle$ and $|2\rangle$.

(b) Now assume we want to cool down the qubit S, and bring it as close to its ground state $|0\rangle\langle 0|_S$ as possible. The only operation we are allowed to perform is a swap operation with one of the virtual qubits from (a); the qutrit V is reset after each iteration. Describe the optimal way to cool down the qubit S.

Exercise 3. Composing virtual qubits

In the lecture, we have seen how to compose two virtual qubits. Now let us repeat the procedure for three.

Suppose that we consider three systems A, B and C. We pick a single virtual qubit within A with the energy gap ΔE^A, a single virtual qubit within B with the energy gap ΔE^B, and a single virtual qubit within C with the energy gap ΔE^C. Assume without loss of generality that $\Delta E^A > \Delta E^B > \Delta E^C$.

(a) For some arbitrary block-diagonal states ρ_A, ρ_B, ρ_C, write down the virtual temperatures of each qubit individually.

(b) Write down the energy eigenstates of the composite system $\rho_A \otimes \rho_B \otimes \rho_C$. Identify which pairs of states give rise to non-local virtual qubits (different from the ones attained if we combine only two of them, or the initial virtual qubits themselves), and calculate their virtual temperatures.

(c) How many (new) virtual qubits can we extract from a composition of n virtual qubits?

(d) Now take m copies of the virtual qubit we chose in A and n copies of the virtual qubit in B and calculate the virtual temperature of the pair $|0\rangle_A^{\otimes m} \otimes |1\rangle_B^{\otimes n}$ and $|1\rangle_A^{\otimes m} \otimes |0\rangle_B^{\otimes n}$.