Exercise 1. Majorization: examples and properties

(a) *(triangles)* Let $\theta_1, \theta_2, \theta_3$ be the angles of a triangle, expressed in radians. Show that

$$(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}) \prec (\theta_1, \theta_2, \theta_3) \prec (\pi, 0, 0).$$

(b) *(measurement outcomes)* Let ρ be an arbitrary state of a d-dimensional quantum system. Prove that there always exists an orthonormal basis $|e_k\rangle$ such that the probabilities for a measurement in that basis are uniformly distributed. Given ρ, can you explicitly construct a basis $|e_k\rangle$ such that this is true?

Hint: Use Schur-Horn theorem.

(c) *(post-measurement state)* Let ρ be a density matrix, and $\{P_j\}$ a complete set of orthonormal projectors. Then the posterior density matrix

$$\rho' = \sum_j P_j \rho P_j$$

satisfies $\rho' \prec \rho$.

(d) *(doubly stochastic matrices)* Show that if D and E are doubly stochastic, then DE is also doubly stochastic. Find an example of vectors r and y and a doubly stochastic matrix D such that $r \prec y$ but $Dr \not\prec Dy$.

(e) *(thermal states)* Compare two thermal states $\tau(\beta_1)$ and $\tau(\beta_2)$ with $\beta_1 < \beta_2$. Which state majorizes the other?

(f) *(energy)* Show that $\rho_1 \succ \rho_2$, and ρ_1 and ρ_2 are passive, then ρ_1 has lower average energy: $E_1 < E_2$.

Exercise 2. Yet again: optimal cooling

Suppose that we are given a (real) qubit S with Hamiltonian $H_S = E_S|1\rangle\langle 1|_S$. Suppose that we also have access to a machine B – an d_B-dimensional system with the Hamiltonian $H_B = \sum_{i=0}^{d_B-1} \mathcal{E}_i|\mathcal{E}_i\rangle\langle\mathcal{E}_i|_B$, where the eigenvalues are ordered: $\mathcal{E}_i \leq \mathcal{E}_j$ for $i < j$; we denote $\mathcal{E}_{d_B-1} = \mathcal{E}_{max}$. The initial states of the machine and the qubit are thermal ones at the inverse temperature β: $\rho_S = \tau_S[\beta]$, $\rho_B = \tau_B[\beta]$. Assume that $\mathcal{E}_{max} > E_S$, and $\beta > 0$.

As before, we want to make the qubit S cooler (no pun intended).

(a) Recall the setting from the previous exercise sheet. There, we have repeatedly performed a swap operation with one of the virtual qubits of the system B, resetting B after each round. Now apply the reasoning to this exercise. Swapping with which virtual qubit of B would lead to the real qubit having the lowest temperature possible, in the limit of the number of operations applied $n \to \infty$? What is this temperature β^*?

Hint: $\beta^* = \frac{\beta\mathcal{E}_{max}}{E_S}$.

(b) The coolest achievable state of the qubit is $\rho_S^* = \tau_S[\beta^*]$. Show that it majorizes the initial state of the qubit.

Moreover, one can show that the following holds: for any unitary U, and any single coherent operation C defined as

$$C(\rho_S) = \mathrm{tr}_B(\ U(\rho_S \otimes \tau_B[\beta])U^\dagger\ ),$$

$C(\rho_S) \prec \rho_S^*$ (you don't need to prove this here, just convince yourselves that it holds :)).

But back to cooling: even though the operation in (a) achieves the cooling bound, it is, in fact, not optimal. Let us consider another operation instead: given a state ρ_{SB}, let U_{opt} be the unitary that reorders the eigenvalues of ρ_{SB} as largest in the energy subspace $|0\mathcal{E}_0\rangle\langle 0\mathcal{E}_0|_{SB}$, second largest in $|0\mathcal{E}_1\rangle\langle 0\mathcal{E}_1|_{SB}$, and so on up to $|1\mathcal{E}_{max}\rangle\langle 1\mathcal{E}_{max}|_{SB}$:

$$U_{opt}\ \rho_{SB}\ U_{opt}^\dagger = \sum_{i=0}^{d_B-1} \left([\rho_{SB}]_i\ |0\mathcal{E}_i\rangle\langle 0\mathcal{E}_i|_{SB}\ +\ [\rho_{SB}]_{d_B+i}\ |1\mathcal{E}_i\rangle\langle 1\mathcal{E}_i|_{SB}\right),$$

where $[\rho_{SB}]_i$ are entries of the vector of the diagonal entries of ρ_{SB}, arranged in non-increasing order.

Now we are interested in the operation $A(\rho_S)$:

$$A(\rho_S) = \mathrm{tr}_B(\ U_{opt}(\rho_S \otimes \tau_B[\beta])U_{opt}^\dagger\ ).$$

(c) **(*)** Show that A is optimal: for any C, $A^{\cdot n}(\rho_S) \succ C^{\cdot n}(\rho_S)$.

Hint: Use that if $\rho_1 \succ \rho_2$, then $\rho_1 \otimes \tau_B[\beta] \succ \rho_2 \otimes \tau_B[\beta]$. By induction, follow the steps:

1. *prove the statement for $n = 1$: $A(\rho_S) \succ C(\rho_S)$;*
2. *prove $\rho_1 \succ \rho_2 \Rightarrow A(\rho_1) \succ C(\rho_2)$;*
3. *assume $A^{\cdot n}(\rho_S) \succ C^{\cdot n}(\rho_S)$, and show $A^{\cdot(n+1)}(\rho_S) \succ C^{\cdot(n+1)}(\rho_S)$.*

(d) **(*)** Show that $A^\infty(\rho_S) \lim_{n\to\infty} A^n(\rho_S) = \rho_S^*$. Thus, we conclude that A is the optimal (fastest) operation which achieves the cooling bound.