Exercise 1. Lorenz curves

In this exercise, we will see that Lorenz curves is a useful tool for illustrating the majorization concept for different states.

(a) Consider a six dimensional system which composed of a qubit S and a qutrit R. We will look at two states: the first state ρ_1 is a composition of a pure state of a qubit $|\psi\rangle_S$ and the maximally mixed state of a qutrit

$$\rho_1 = |\psi\rangle\langle\psi|_S \otimes \frac{1}{3}\mathbb{1}_R;$$

and the second state ρ_2 is a composition of a pure state of a qutrit $|\phi\rangle_R$ and the maximally mixed state of the qubit

$$\rho_2 = \frac{1}{2}\mathbb{1}_S \otimes |\phi\rangle\langle\phi|_R.$$

Draw Lorenz curves for both states. What is the difference?

(b) Lorenz curves are useful not only in quantum thermodynamics; moreover, they were invented for a completely different purpose. Assume that the probability distribution is the income distribution: the percentage of GDP (gross domestic product) that constitutes the income of a member of the population. Which curves correspond to perfect (economic) equality and perfect inequality?

Hint: Not a hint, but a suggestion: search for the Lorenz curves of different countries. This is surprisingly a very illustrative way to assess how economically equal the society is!

Exercise 2. Sets of noisy operations

In this exercise, we investigate the structure of two sets: the set of noisy classical operations, and the set of noisy quantum operations. For completeness, we also list the definitions of all operations mentioned.

Definition 1 (Noisy classical operation)**.** *A noisy classical operation is a positivity-preserving and normalization-preserving map $D : V_{in} \to V_{out}$ that admits the following decomposition: there exists an ancilla system with a discrete physical space Ω_a and a permutation on $\Omega_{in} \times \Omega_a$ with an induced representation π on $V(\Omega_{in}) \otimes V(\Omega_a)$ such that for all input states x_{in}*

$$Dx_{in} = \sum_{\Omega_{a'}} \pi(x_{in} \otimes m_a),$$

where m_a is the normalized uniform distribution on Ω_a, and $\Omega_{a'}$ is the physical state space complementary to Ω_{out}: $\Omega_{in} \times \Omega_a = \Omega_{out} \times \Omega_{a'}$.

Definition 2 (Uniform-preserving stochastic map)**.** *A stochastic map D is uniform-preserving if it takes a uniform distribution on the input system to the uniform distribution on the output system.*

(a) What are the properties of the uniform-preserving stochastic matrices? What can you say about the case when the input and output spaces are of equal dimension $d_{in} = d_{out}$?

(b) Prove that the set of noisy classical operations coincides with the set of uniform-preserving stochastic matrices and, in the case of equal dimension of input and output spaces, it coincides with the set of mixtures of permutations.

Now let us turn to the noisy quantum operations.

Definition 3 (Noisy quantum operation)**.** *A noisy quantum operation $\mathcal{E}$ is one that admits the following decomposition: there exist a finite-dimensional ancilla space $\mathcal{H}_a$ and a unitary U on $\mathcal{H}_{in} \otimes \mathcal{H}_a$ such that for all input states ρ_{in}*

However, the relationship between the set of unital operations and the set of quantum noisy operations is not as straightforward as for their classical counterparts. Noisy quantum operations form a strict subset of the unital operations, and, in the case of equal dimension of input and output space, a strict superset of the mixtures of unitaries.

(c) Prove that a noisy quantum operation is necessarily unital.

(d) Prove that a mixture of unitaries is necessarily a noisy quantum operation.

Hint: Given an ensemble of unitaries (p_i, U_i), consider an ancilla (of an arbitrarily large dimension) in the completely mixed state, and partition its Hilbert space into subspaces, the relative dimensions of which are described by the distribution $\{p_i\}$.

Proving strictness of the inclusion is more complicated: example of a unital operation that is not a quantum noisy operation is provided in (Haagerup and Musat, 2011), and the fact that not every quantum noisy operation is a mixture of unitaries has been shown by Shor in (Shor, 2010).

References:

Haagerup, U., and M. Musat, 2011, Commun. Math. Phys. 303(2), 555.

Shor, P. W., 2010, Structure of Unital Maps and the Asymptotic Quantum Birkhoff Conjecture, presentation.

Exercise 3. Thermal bath heat capacity

Heat capacity of a system can be defined as the rate of change of inverse temperature with adding heat, $\kappa = d\beta/dQ$.

(a) Calculate the heat capacity for a single system in a thermal state $\tau[\beta]$.

One way to arrive to the thermal bath is to take n copies of a system in a thermal state:

$$\rho = \tau[\beta]^{\otimes n}.$$

(b) Show that the heat capacity of this composite system scales as n^{-1}.