Exercise 1. Carnot engine as a source of work

Here we will look at the derivation of the Landauer's bound for the case when we model the work source as a Carnot engine consisting of two baths. But first, let us recap the case with just one bath.

Suppose that the setting consists of two finite-dimensional systems, the target system S and the machine M. The initial state is an uncorrelated one, with the machine in a thermal state of temperature $\beta \geq 0$ and Hamiltonian H_M. Let U a global unitary we act on both systems; we label the final joint state as ρ'_{SM}, and the respective reduced states as ρ'_S and ρ'_M. We will need two more quantities: the final mutual information between systems S and M

$$I(S:M)_{\rho'_{SM}} = S(\rho'_S) + S(\rho'_M) - S(\rho'_{SM}),$$

and the relative entropy of the final state of the machine with respect to its initial state

$$D(\rho'_M||\rho_M) = \mathrm{tr}(\rho'_M \log \rho'_M) - \mathrm{tr}(\rho'_M \log \rho_M).$$

(a) Show that

$$[S(\rho'_S) - S(\rho_S)] + [S(\rho'_M) - S(\rho_M)] = I(S:M)_{\rho'_{SM}} \geq 0.$$

(b) Using the result from (a), show the equality form of Landauer's principle

$$\beta\,\mathrm{tr}\left(H_M(\rho'_M - \rho_M)\right) - [S(\rho'_S) - S(\rho_S)] = I(S:M)_{\rho'_{SM}} + D(\rho'_M||\rho_M) \geq 0.$$

How does the usual inequality form of Landauer's principle follow?

Now let us consider an incoherent setting: suppose that the machine is composed of two parts: H coupled to a hot bath of temperature β_H, and R coupled to a cold bath of temperature β_R, $\beta_R \geq \beta_H \geq 0$. The global unitary U is assumed to be energy-preserving

$$[U, H_S + H_H + H_R] = 0.$$

Let us further introduce the following notation:

- The entropy change on a subsystem: $\Delta S_\cdot = S(\rho'_\cdot) - S(\rho_\cdot)$;
- The energy change on a subsystem: $\Delta E_\cdot = \mathrm{tr}\left(H_\cdot(\rho'_\cdot - \rho_\cdot)\right)$;
- The free energy of a state ρ at temperature β: $F^{(\beta)}(\rho) = \mathrm{tr}(H\rho) - \frac{1}{\beta}S(\rho)$ (its change is labeled as $\Delta F^{(\beta)}$);
- The Carnot efficiency of a machine with temperatures β_H and β_R: $\eta = 1 - \frac{\beta_H}{\beta_R}$.

The initial state ρ_{SRH} is uncorrelated between S, C and H.

(c) Show that the following equality holds:

$$-\Delta F_S^{(\beta_R)} - \eta \Delta E_H = \frac{1}{\beta_R}\left(\Delta S_S + \Delta S_R + \Delta S_H + D(\rho'_R||\rho_R) + D(\rho'_H||\rho_H)\right) \geq 0.$$

How does this equality relate to the original Landauer's principle?

Hint: The mutual information is always non-negative due to subadditivity of entropy. Apply this observation to the final mutual information between S, C and H:

$$I(S:R:H)_{\rho'_{SRH}} = S(\rho'_S) + S(\rho'_R) + S(\rho'_H) - S(\rho'_{SRH}) \geq 0.$$

(d) Now assume that the target system S is actually a cold bath of temperature β_S, $\beta_S > \beta_R > \beta_H$. We seek to extract energy from S (thus cooling it further) by using the heat flowing from H to R. Show that in this case, the second law involves the Carnot efficiency of an absorption refrigerator.