

Exercise 1. Lindbladian evolution

Lindbladian evolution is an extension of Schrödinger's equation to open systems. In this exercise, we will walk through necessary steps to derive it.

- (a) Write down the evolution of a state $\rho(t)$ with Hamiltonian H over a tiny period of time δt . Use Taylor expansion for the exponential, and express $\rho(t + \delta t)$ in terms of $\rho(t)$ up to the first order in δt . Take the limit $\delta t \rightarrow 0$. This is the time-independent Schrödinger equation for mixed states.

Next we consider the notion of open systems: the concept of “openness” in general physics implies that the system can interact with an external environment. In order to schematize this in quantum theory, we consider a Hilbert space of the form:

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

where S is our open system and E represents the environment. Generally, we assume that the environment is a much larger system: $\dim \mathcal{H}_S \ll \dim \mathcal{H}_E$. To obtain the dynamics of the joint system, we need to solve the Schrödinger equation:

$$H_{SE} \mapsto U_{SE}(t) = \exp\left(-\frac{it}{\hbar} H_{SE}\right)$$

However, we may not even know all the details of the interaction enough to approximate the Hamiltonian. On the other hand, we do not need to keep track of the evolution of the state ρ_{SE} of the whole system, in particular it suffices for us to know enough of the state of the system ρ_S , i.e. the partial trace

$$\rho_S(t) = \text{tr}_E \left(U_{SE}(t) \rho_{SE}(0) U_{SE}^\dagger(t) \right)$$

We can also describe the operation as a TPCPM \mathcal{E} acting on the system S alone, which approximates what happens in the environment:

$$\rho_S(t) = \mathcal{E}_S(t, \rho_S(0))$$

In order for this approximation to be good enough, we assume that the dynamics of the environment is much faster than the interaction, which means that previous correlations between the system and the environment become negligible.

- (b) Consider a variation $\delta\rho$, $|\delta\rho| \ll |\rho|$:

$$\mathcal{E}(\delta t, \rho_S(t)) \simeq \rho(t) + \delta\rho$$

Write down the Kraus decomposition of the channel. Let us rewrite them in the following way:

$$A_0 = \mathbb{1} + \delta t(L_0 - iK)$$

$$A_k = L_k \sqrt{\delta t}, \quad k > 0$$

where K and L_k for every k are bounded operators. Analyze the terms of the sum in the Kraus decomposition.

(c) Letting $\delta t \rightarrow 0$, show that

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H_S, \rho] + \{L_0, \rho\} + \sum_{k=1}^{\infty} L_k \rho L_k^\dagger.$$

Hint: Choose $K = H_S/\hbar$, using intuition from (a).

(d) Suppose that S and E don't interact. Determine L_k in this case.

(e) Using that the trace of the density matrix is always preserved $\text{tr}\left(\frac{d\rho}{dt}\right) = 0$, find the expression for L_0 in terms of all other L_k , $k \neq 0$.

Exercise 2. Generating entanglement with a thermal machine [1]

We consider two qubits with identical energy gaps E weakly coupled to each other and to separate thermal reservoirs. We denote the ground and excited states $|0\rangle$, $|1\rangle$, and the free Hamiltonian for the qubits in this basis is

$$\hat{H}_0 = E(|1\rangle\langle 1| \otimes \mathbb{1} + \mathbb{1} \otimes |1\rangle\langle 1|),$$

The interaction Hamiltonian, which is energy conserving, is given by

$$\hat{H}_{int} = g(|10\rangle\langle 01| + |01\rangle\langle 10|),$$

where g is the strength of the coupling between the qubits. The coupling to the thermal baths is modelled using a simple collision model, where thermalisation happens through rare but strong events. At every time step, each qubit k is either reset to a thermal state τ_k at the temperature of its bath with a small probability or left unchanged. The state of the qubits evolves according to the master equation

$$\frac{\partial \rho}{\partial t} = i[\rho, \hat{H}_0 + \hat{H}_{int}] + \sum_{k \in \{c, h\}} p_k (\Phi_k(\rho) - \rho)$$

where p_k is the thermalisation rate for qubit k and $\Phi_c(\rho) = \tau_c \otimes \text{tr}_c \rho$ and $\Phi_h(\rho) = \text{tr}_h \rho \otimes \tau_h$. We take the first qubit to have the colder and the second to have the warmer bath temperature. We refer to them as the 'cold' and 'hot' qubit respectively and use subscripts c and h . The thermal states are given by $\tau_k = r_k |0\rangle\langle 0| + (1 - r_k) |1\rangle\langle 1|$ with occupation probabilities determined by the Boltzmann factor according to $r_k = 1/(1 + e^{-E/T_k})$ where T_k is the reservoir temperature for qubit k (throughout the paper we set $k_B = 1$ and $\hbar = 1$).

Note: the master equation applies only in the perturbative regime $p_c, p_h, g \ll E$ and $p_c, p_h \ll 1$.

Here, we want to obtain the steady state solution of the master equation.

(a) Check that the following is a steady solution of the master equation:

$$\rho_\infty = \gamma \left[p_c p_h \tau_c \otimes \tau_h + \frac{2g^2}{(p_c + p_h)^2} (p_c \tau_c + p_h \tau_h)^{\otimes 2} + \frac{g p_c p_h (r_c - r_h)}{p_c + p_h} \mathcal{Y} \right]$$

with $\mathcal{Y} = i|01\rangle\langle 10| - i|10\rangle\langle 01|$ and $\gamma = 1/(2g^2 + p_c p_h)$, and where $\rho^{\otimes 2} = \rho \otimes \rho$. Note that for resonant qubits, the steady state depends on the energy E only through r_c, r_h .

- (b) Let us now determine how entangled the steady state is. To quantify this, use the notion of concurrence [2], which is an entanglement monotone defined for a mixed state of two qubits as:

$$\mathcal{C}(\rho) \equiv \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues (in decreasing order) of the Hermitian matrix $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

- (c) Consider the cases with $T_c = T_h$, and $T_h \rightarrow \infty$ and $T_c = 0$.

References

- [1] Jonatan Bohr Brask, Nicolas Brunner, Géraldine Haack, Marcus Huber, *Autonomous quantum thermal machine for generating steady-state entanglement*. **New Journal of Physics** **17**, **113029** (2015). [arXiv:1504.00187](https://arxiv.org/abs/1504.00187).
- [2] William K. Wootters, *Entanglement of Formation of an Arbitrary State of Two Qubits*. **Physical Review Letters**, **10(80)** (1998). DOI: [10.1103/physrevlett.80.2245](https://doi.org/10.1103/physrevlett.80.2245).