Exercise 1. The dynamics of the ideal clock

Let us consider an idealized clock: a system C with the Hamiltonian $\hat{H} = \hat{p}_c$.

(a) What is the distinguishable basis of the time states for such a Hamiltonian? Write down the natural evolution of the clock. What does a time translation physically correspond to?

(b) Show that this clock is "continuous": the temporal and spatial translations are equivalent for any state of the clock.

(c) Let us add a position-dependent potential $V(x_c)$ (which happens to be an infinitely differentiable function of compact support on $L_2(\mathbb{R})$) to the clock. Show that the clock still remains continuous, only gaining a phase from the potential

$$\langle x|e^{-it\left(\hat{H}+V(\hat{x}_c)\right)}|\psi_c\rangle = e^{-i\int_{x-t}^{x} V(x')dx'}\langle x-t|\psi_c\rangle, \quad x,t \in \mathbb{R}.$$

Now take a step back, and consider a system S in a state $\rho_s \in \mathcal{S}(\mathcal{H}_s)$, upon which one wishes to perform the (energy-preserving) unitary U_s over a time interval $[t_i, t_f]$. The unitary can be implemented by the addition of $\hat{H}_s^{int}$ as a time-dependent interaction

$$\hat{H} = \hat{H}_s + \hat{H}_s^{int} \cdot V(t),$$

where $V(t)$ is a normalized pulse, i.e.

$$\int_{t_i}^{t_f} dt V(t) = 1.$$

To get rid of the explicit time-dependence of the Hamiltonian, we add a second system: a clock, and replace the background time parameter t by the time degree of the clock. For the idealised clock, this corresponds to a Hamiltonian on $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_c$, given by

$$\hat{H}_{total}^{id} = \hat{H}_s \otimes \mathbb{1}_c + \mathbb{1}_s \otimes \hat{p}_c + \hat{H}_s^{int} \otimes V(\hat{x}_c).$$

(d) Take the initial state of the system and clock be in a product form

$$|\psi_{sc}^0\rangle = |\psi_s^0\rangle \otimes |\psi_c^0\rangle,$$

and assume that $H_s = 0$. Verify that this Hamiltonian can indeed implement the unitary U_s.

Exercise 2. The limit of p Hamiltonian

In this exercise, we will look at various limits of the finite clock Hamiltonian

$$H_c = \sum_{n=0}^{d-1} n\omega |n\rangle\langle n|_c.$$

(a) First show that the recurrence time of the clock is $T = \frac{2\pi}{\omega}$.

(b) Let us first consider $d \to \infty$, with $\omega d \to \infty$. What is the limit of the Hamiltonian in this case? What is the physical meaning of the limit?

(c) Now suppose that $\omega d = const$. How does your answer from (b) change?