Exercise 1. The quantum pigeonhole principle [1]

It is a truth universally acknowledged, that a person in a possession of three pigeons and two holes, will observe one hole occupied by at least two pigeons. However, this is not true for pigeons who are under the command of quantum mechanics. In this exercise, we will consider instances when three quantum particles are put in two boxes, yet no two particles are in the same box, generalize the result, and discuss implications of the result for our understanding of quantum correlations.

Consider three particles and two boxes, denoted L (left) and R (right). To start our experiment, we prepare each particle in a superposition of being in the two boxes,

$$|\Psi\rangle_j = \frac{1}{\sqrt{2}}\left(|L\rangle_j + |R\rangle_j\right),\ j \in \{1,2,3\}.$$

(a) Write down projectors corresponding to the subspaces of: particles 1 and 2 being in the same box $\Pi_{1,2}^{\text{same}}$; and them being in different boxes $\Pi_{1,2}^{\text{diff}}$. What are the probabilities of these events given the initial state $|\Psi\rangle = |\Psi\rangle_1|\Psi\rangle_2|\Psi\rangle_3$?

Now, given the same initial state $|\Psi\rangle$, let us measure whether each particle is in the state $|{+i}\rangle = \frac{1}{\sqrt{2}}(|L\rangle + i|R\rangle)$ or $|{-i}\rangle = \frac{1}{\sqrt{2}}(|L\rangle - i|R\rangle)$. The cases we are interested in are those in which all particles are found in $|{+i}\rangle$, i.e. the final state

$$|\Phi\rangle = |{+i}\rangle_1|{+i}\rangle_2|{+i}\rangle_3.$$

Neither the initial state nor the finally selected state contain any correlations between the position of the particles. Furthermore, both the preparation and the post-selection are done independently, acting on each particle separately.

(b) What are the probabilities of finding the particles 1 and 2 in the same or different boxes at the intermediate time, given the final (post-selected) state $|\Phi\rangle$, and the initial (pre-selected) state $|\Psi\rangle$? Explain why the result can be applied to all pairs of particles.

To sum up, given the above pre- and post-selection, we have three particles in two boxes, yet no two particles can be found in the same box – our quantum pigeonhole principle.

(c) How the principle changes if we post-select on the states $|\Phi\rangle = |{-i}\rangle_1|{-i}\rangle_2|{-i}\rangle_3$ or $|\Phi\rangle = |{+i}\rangle_1|{-i}\rangle_2|{-i}\rangle_3$?

(d) Now consider N particles in M boxes. Show that we can guarantee that no two particles are in the same box when we prepare each particle in the state

$$|0\rangle_j = \frac{1}{\sqrt{M}}\sum_{k=1}^{M}|k\rangle_j,\ j \in \{1,\ldots,N\},$$

and post-select on each particle being in the state

$$|\eta\rangle_j = \frac{1}{\sqrt{M}}\sum_{k=1}^{M} e^{\frac{i\pi k}{M}}|k\rangle_j,\ j \in \{1,\ldots,N\}.$$

(e) Coming back to the case of three particles and two boxes, compare the probabilities of finding two particles in the same box for the "pre- selected only" experiment, and for the "pre- and post- selected" one. What does the conclusion tell you about the difference between correlations that can be observed when we measure particles separately and when we measure them jointly?

References

[1] Y. Aharonov, F. Colombo, S. Popescu, I. Sabadini, D. C. Struppa, J. Tollaksen, *The quantum pigeonhole principle and the nature of quantum correlations*, 2014. arXiv:1407.3194.