Exercise 1. Counterfactuals in classical epistemic logic

Counterfactual statements, or *counterfactual modalities*, concern what is not, but could have been. As an example, consider answers to the following questions: what would have happened if the pandemic didn't spread across the world in 2020? What if I were to take a holiday this week or the next?

Counterfactuals, as we have seen in the lecture, are also used in a variety of quantum thought experiments. Generally, these are simply statements expressing a particular logical inference, $a \Rightarrow b$: "from a it follows that b", "no pandemic $\Rightarrow$ classes in person". Counterfactual reasoning (in classical setting) is also used to solve logical puzzles. In this exercise, we will consider two of those.

(a) Three logicians, Alice, Bob, and Charlie, walk into a bar. Alice asks, "Does everyone want a beer?" Bob answers, "I don't know." Charlie answers "I don't know" as well. Alice smiles, and orders beer for all three logicians. Explain her reasoning.

(b) (*Sleeping Beauty problem [1]*) Sleeping Beauty volunteers to undergo the following experiment and is told all of the following details: On Sunday she will be put to sleep. Once or twice, during the experiment, Sleeping Beauty will be awakened, interviewed, and put back to sleep with an amnesia-inducing drug that makes her forget that awakening. A fair coin will be tossed to determine which experimental procedure to undertake:

- if the coin comes up heads, Sleeping Beauty will be awakened and interviewed on Monday only;
- if the coin comes up tails, she will be awakened and interviewed on Monday and Tuesday.

Any time Sleeping Beauty is awakened and interviewed she will not be able to tell which day it is or whether she has been awakened before. During the interview Sleeping Beauty is asked: "What is your probability assignment now for the proposition that the coin landed heads?" You are the Sleeping Beauty; give your answer.

Exercise 2. Measurements: from von Neumann to weak

After carrying out a quantum measurement, we obtain a (classical) outcome. As a piece of information, this outcome is physical, and can be modelled as being contained in a physical system. In this exercise, we take this observation into account to arrive to von Neumann picture of measurement [2].

Suppose we want to measure an observable $\hat{A}$ with the basis $\{|a_i\rangle\}_i$ on a system S. Consider another system (a test particle) M described by the canonical position $\hat{X}$ and conjugate momentum $\hat{P}$ which we couple the system S to via the interaction Hamiltonian

$$H_I = g(t)\hat{A}_S \otimes \hat{P}_M.$$

The time dependent coupling constant $g(t)$ describes the switching "on" and "off" of the interaction. For an impulsive measurement we need the coupling to be strong and short; we take $g(t)$ to be non-zero only for a short time around the moment of interest, t_0 and such that $\int g(t)dt = g > 0$. We assume that during the time of measurement the evolution is governed solemnly by the interaction term.

(a) The initial state of the measured system and the measurement device (the pointer) is

$$|\chi\rangle_{SM} = \sum_i \alpha_i |a_i\rangle_S \otimes \int dx \ \psi(x)|x\rangle_M.$$

Write down the final state of the system after the measurement.

(b) Let us take a Gaussian as the initial state of the pointer $\psi(x) = \exp(-\frac{x^2}{\Delta^2})$. When does this measurement approach an ideal one?

Hint: For an ideal measurement, the final state of the pointer (after tracing out the system S) is a density operator representing a series of peaks, each corresponding to a different eigenvalue a_i, with probability $|\langle a_i|\psi\rangle|^2$.

(c) We can reduce the disturbance caused by the measurement on the measured system by reducing the strength of the interaction g. What is the approximate final state of the system M in that case?

(d) Now let us add the pre- and post-selection on the system S to the picture. Write down the final state of the measuring device, given the initial state $|\Psi\rangle_S$ and the final state $|\Phi\rangle_S$ of the system S.

It can be shown [3] that the final state of the measuring device after post-selection is described by the Gaussian $\exp(-\frac{(x-g\,A_w)^2}{\Delta^2})$, which corresponds to the pointer being shifted to a value A_w, called the *weak value* of the observable $\hat{A}$, and given by

$$A_w = \frac{\langle\Phi|\hat{A}|\Psi\rangle}{\langle\Phi|\Psi\rangle}.$$

Note that in contrast to ordinary expectation values, weak values can lie outside the range of eigenvalues of $\hat{A}$ and are generally complex!

Most importantly, in the weak regime different measurements do not disturb each other so non-commuting variables $\hat{A}$ and $\hat{B}$ can be measured simultaneously, and yield the same weak values A_w and B_w as when measured separately.

References

[1] for example, `https://en.wikipedia.org/wiki/Sleeping_Beauty_problem`

[2] John von Neumann. Mathematical foundations of quantum mechanics. *Princeton University Press, Princeton, 1953.*

[3] Yakir Aharonov, Lev Vaidman. Properties of a quantum system during the time interval between two measurements. *Physical Review A, 41(1), 11–20*, 1990. DOI:10.1103/physreva.41.11