Exercise 1. Pre- and post-selection paradoxes (based on [1])

Suppose a quantum system is prepared in state $|\psi\rangle$, subjected to an intermediate projective measurement $\mathcal{M} = \{P_j\}$, followed by a final projective measurement that includes the projector onto $|\phi\rangle$ as one of its outcomes. Assuming that no other evolution occurs, the joint probability for obtaining the outcome P_j and passing the post-selection is

$$\mathbb{P}(P_j, \phi|\psi, \mathcal{M}) = |\langle\phi|P_j|\psi\rangle|^2, \tag{1}$$

(a) Write down the marginal probability for passing the post-selection $\mathbb{P}(\phi|\psi, \mathcal{M})$. Show that the probability for the intermediate measurement conditioned on both the pre- and post-selection can be written as

$$\mathbb{P}(P_j|\psi, \mathcal{M}, \phi) = \frac{\mathbb{P}(P_j, \phi|\psi, \mathcal{M})}{\mathbb{P}(\phi|\psi, \mathcal{M})} = \frac{|\langle\phi|P_j|\psi\rangle|^2}{\sum_k |\langle\phi|P_k|\psi\rangle|^2}. \tag{2}$$

Here we shall only consider cases $\mathbb{P}(P|\psi, \{P, \mathbb{1} - P\}, \phi)$ where the intermediate measurement has two outcomes, and abbreviate $\mathbb{P}(P|\psi, \{P, I-P\}, \phi) = \mathbb{P}(P|\psi, \phi)$ and $\mathbb{P}(\phi|\psi, \{P, I-P\}) = \mathbb{P}(\phi|\psi)$.

Now let use define a logical pre- and post-selection (PPS) paradox. Consider a Hilbert space, a choice of pre-selection $|\psi\rangle$ and post-selection $|\phi\rangle$, and a (finite) set of projectors $\mathcal{P}$ that is closed under complements, i.e. if $P \in \mathcal{P}$ then $\mathbb{1} - P \in \mathcal{P}$. Suppose further that the probabilities $\mathbb{P}(P|\psi, \phi)$ are either 0 or 1 for every $P \in \mathcal{P}$ (which is what leads to the terminology "logical").

Now consider the partial boolean algebra generated by $\mathcal{P}$, i.e. the smallest set of projectors $\mathcal{P}'$ that contains $\mathcal{P}$ and satisfies

- If $P \in \mathcal{P}'$ then $\mathbb{1} - P \in \mathcal{P}'$.
- If $P, Q \in \mathcal{P}'$ and $PQ = QP$ then $PQ \in \mathcal{P}'$.

If we think of projectors as representing propositions, then these conditions ensure that we can take complements and conjunctions of compatible propositions.

Finally, suppose that we try to extend the probability function $f(P) = \mathbb{P}(P|\psi, \phi)$ from $\mathcal{P}$ to $\mathcal{P}'$ such that the following *algebraic conditions* are satisfied

(i) For all $P \in \mathcal{P}'$, $0 \leq f(P) \leq 1$.

(ii) $f(I) = 1, f(0) = 0$.

(iii) For all $P, Q \in \mathcal{P}'$ such that $PQ = QP$, $f(P + Q - PQ) = f(P) + f(Q) - f(PQ)$.

If it is not possible to do this then we say that the ABL predictions for $\mathcal{P}$ form a *logical PPS paradox.*

(b) Show that the quantum pigeonhole principle is a logical PPS paradox.

(c) Show that Hardy's paradox is also an example of a logical PPS paradox.

(d) Consider another setting: a so-called three-box paradox. It involves a state space spanned by $\{|1\rangle, |2\rangle, |3\rangle\}$ representing a ball in box 1, 2, or 3 respectively. Consider a pre-selection $|\psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$ and a post-selection $|\phi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - |3\rangle)$. Compare the probabilities of finding the ball in the 1st and 2nd boxes. Show that this is also a case of a PPS paradox.

Exercise 2. Pre- and post-selection paradoxes with weak values

As we have established previously, an observable A can be measured by coupling the system to a continuous variable pointer system via a Hamiltonian $H = gA \otimes p$, where g is the coupling constant, A is the observable to be measured, and p is the momentum of the pointer. If the parameters are chosen such that $gt \ll \Delta x$, where t is the duration of the measurement interaction and Δx is the initial position uncertainty of the pointer, then this is called a "weak measurement".

If the system is pre- and post-selected, with a weak measurement in the middle, then, to first order in gt, the position distribution of a suitably prepared pointer simply shifts by an amount $gtw(A|\psi, \phi)$, where

$$w(A|\psi, \phi) = \mathrm{Re}\left(\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}\right), \tag{3}$$

and $w(A|\psi, \phi)$ is called the *weak value* of A. Weak values can lie outside the eigenvalue range of the operator A, in which case they are called *anomalous* weak values.

(a) Check that the weak values assigned to a partial boolean algebra of projection operators always satisfy the algebraic conditions in Exercise 1 with $f(P) = w(P|\psi, \phi)$.

(b) Verify $\mathbb{P}(P|\psi, \phi)$ is 0 or 1 then $w(P|\psi, \phi) = \mathbb{P}(P|\psi, \phi)$. What does this mean for weak values of weak measurement versions of logical PPS paradoxes?

References

[1] Matthew F. Pusey, Matthew S. Leifer. Logical pre- and post-selection paradoxes are proofs of contextuality. *EPTCS 195, pp. 295-306 (2015).* arXiv:1506.07850