Exercise 1. ***Cooling a real qubit with a virtual one***

In this exercise, we consider the process of cooling the qubit by repeatedly swapping it with a resetted virtual qubit. Let S be a real qubit which is initially in the fully mixed state:

$$\rho_S = \frac{1}{2}|0\rangle\langle 0|_S + \frac{1}{2}|1\rangle\langle 1|_S.$$

Our goal is to cool this qubit to its ground state $|0\rangle\langle 0|_S$. For this we can use a larger dimensional system V, and choose a virtual qubit with temperature $\beta_V = +\infty$ (all population is concentrated on the level with lower energy) and norm N_V.

(a) What is the reduced state of the qubit S after applying the swapping map n times? The virtual qubit is reset at each iteration.

Solution Making use of the derivation in the lecture notes (Chapter 2.5.3), we arrive to the reduced state of the qubit S after n applications of swap:

$$\rho_S^{(n)} = \tau_S\left[\beta_V\right] + (1 - N_V)^n(\rho_S - \tau_S\left[\beta_V\right]).$$

The virtual temperature $\beta_V = +\infty$, so $\tau_S\left[\beta_V\right] = |0\rangle\langle 0|_S$. Given that the initial state is a fully mixed one, we write

$$\rho_S^{(n)} = \left(1 - \frac{(1 - N_V)^n}{2}\right)|0\rangle\langle 0|_S + \frac{(1 - N_V)^n}{2}|1\rangle\langle 1|_S.$$

(b) How many times do you need to apply the swap to get a state of S, which is ϵ-close to the ground state $\rho_S^{(N)} = (1 - \epsilon)|0\rangle\langle 0|_S + \epsilon|1\rangle\langle 1|_S$, $\epsilon \ll 1$?

Solution From our result in (a) we obtain $\epsilon = \frac{(1-N_V)^N}{2}$, and $N = \frac{\ln(2\epsilon)}{\ln(1-N_V)}$. For a given N_V, N scales logarithmically with respect to ϵ.

(c) Assuming that the energy gaps of the real and virtual qubits are E_S and E_V respectively, calculate the change in the energy of the system after each step. What is the total energy cost of cooling to $|0\rangle\langle 0|_S$?

Solution The energy difference can be written as (here we use the notation of Chapter 2.5.3 for a general case):

$$\Delta E = (1 - p_S)p_i E_V + p_S p_j E_S - (1 - p_S)p_i E_S - p_S p_j E_V = ((1 - p_S)p_i - p_S p_j)(E_V - E_S).$$

In this exercise, $p_i = N_V$, $p_j = 0$, so at the nth step

$$\Delta E^{(n)} = N_V(1 - p_S^{(n)})(E_V - E_S) = N_V(E_V - E_S)\frac{(1 - N_V)^n}{2}.$$

The total energy change:

$$\Delta E = N_V(E_V - E_S)\sum_{n=0}^{\infty}\frac{(1 - N_V)^n}{2} = \frac{E_V - E_S}{2}.$$

Exercise 2. ***Optimize the cooling of qubit with a qutrit***

Let us take a qutrit B as an example of a multi-dimensional system to take virtual qubits from, described by a Hamiltonian

$$H_B = E_B|1\rangle\langle 1|_B + 2E_B|2\rangle\langle 2|_B.$$

The qutrit is initially in the state

$$\rho_B = \frac{2}{3}|0\rangle\langle 0|_B + \frac{1}{4}|1\rangle\langle 1|_B + \frac{1}{12}|2\rangle\langle 2|_B.$$

Let us also consider a qubit S, initially in the fully mixed state.

(a) Characterize two virtual qubits of V: one making use of the eigenstates $|0\rangle$ and $|1\rangle$, and one – of the eigenstates $|1\rangle$ and $|2\rangle$.

Solution The virtual qubit made of the eigenstates $|0\rangle$ and $|1\rangle$:

$$\text{norm: } N_1 = \frac{2}{3} + \frac{1}{4} = \frac{11}{12}$$
$$\text{thermal state: } \tau\left[\beta_1\right] = \frac{8}{11}|0\rangle\langle 0| + \frac{3}{11}|1\rangle\langle 1|.$$

The virtual qubit made of the eigenstates $|1\rangle$ and $|2\rangle$:

$$\text{norm: } N_2 = \frac{1}{4} + \frac{1}{12} = \frac{1}{3}$$
$$\text{thermal state: } \tau\left[\beta_2\right] = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|.$$

(b) Now assume we want to cool down the qubit S, and bring it as close to its ground state $|0\rangle\langle 0|_S$ as possible. The only operation we are allowed to perform is a swap operation with one of the virtual qubits from (a); the qutrit V is reset after each iteration. Describe the optimal way to cool down the qubit S.

Solution Applying a swap operation with the initial state of the qubit ρ_S, and a virtual qubit V maps S to the state

$$\rho'_S = N_V \tau_S\left[\beta_V\right] + (1 - N_V)\rho_S;$$

after resetting the virtual qubit and repeating the operation n times, we arrive to

$$\rho_S^{(n)} = \tau_S\left[\beta_V\right] + (1 - N_1)^n(\rho_S - \tau_S\left[\beta\right]) = p_S^{(n)}|0\rangle\langle 0| + (1 - p_S^{(n)});$$

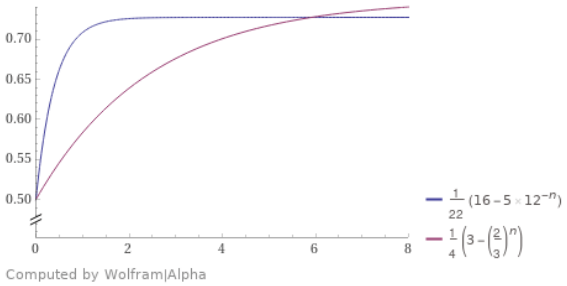
To cool optimally, we need to maximize $p_S^{(n)}$ over the number at each step. Because we have two virtual qubits, let us analyze the cooling for each for them separately, and then compare the performance to see how to combine them optimally.
Using the first virtual qubit, given that $\rho_S = \frac{1}{2}|0\rangle\langle 0|_S + \frac{1}{2}|1\rangle\langle 1|_S$, after n swaps we get:

$$p_S^{(n)}{}_1 = \frac{8}{11} - \frac{5}{22}\left(\frac{1}{12}\right)^n.$$

Using the second virtual qubit, we get

$$p_S^{(n)}{}_2 = \frac{3}{4} - \frac{1}{4}\left(\frac{2}{3}\right)^n.$$

Plotting these two functions against each other:
Comparing the plots, we see that it is more optimal to use the first virtual qubit for $n \approx 2$ swaps, and then switch to using the second one.



Exercise 3. ***Composing virtual qubits***

In the lecture, we have seen how to compose two virtual qubits. Now let us repeat the procedure for three.

Suppose that we consider three systems A, B and C. We pick a single virtual qubit within A with the energy gap ΔE^A, a single virtual qubit within B with the energy gap ΔE^B, and a single virtual qubit within C with the energy gap ΔE^C. Assume without loss of generality that $\Delta E^A > \Delta E^B > \Delta E^C$.

(a) For some arbitrary block-diagonal states ρ_A, ρ_B, ρ_C, write down the virtual temperatures of each qubit individually.

Solution Suppose that the single virtual qubits within (A, B, C) we picked are between the energy eigenstates $|E_i^{(A,B,C)}\rangle$ and $|E_j^{(A,B,C)}\rangle$, with the populations $p_i^{(A,B,C)}$ and $p_j^{(A,B,C)}$, and energy gaps $\Delta E^{(A,B,C)} = E_j^{(A,B,C)} - E_i^{(A,B,C)}$. Then their virtual temperatures are defined by

$$e^{-\beta_v^A \Delta E_A} = \frac{p_j^A}{p_i^A}$$
$$e^{-\beta_v^B \Delta E_B} = \frac{p_j^B}{p_i^B}$$
$$e^{-\beta_v^C \Delta E_C} = \frac{p_j^C}{p_i^C}.$$

(b) Write down the energy eigenstates of the composite system $\rho_A \otimes \rho_B \otimes \rho_C$. Identify which pairs of states give rise to non-local virtual qubits (different from the ones attained if we combine only two of them, or the initial virtual qubits themselves), and calculate their virtual temperatures.

Solution Three virtual qubits, one each from A, B and C, generate an eight dimensional subspace in the composite system. The eigenstates and their eigenvalues of this subspace

are

$$
\begin{aligned}
|000\rangle_{ABC}: &\quad \{|E_i^A\rangle \otimes |E_i^B\rangle \otimes |E_i^C\rangle, \quad E_i^A + E_i^B + E_i^C\} \\
|001\rangle_{ABC}: &\quad \{|E_i^A\rangle \otimes |E_i^B\rangle \otimes |E_j^C\rangle, \quad E_i^A + E_i^B + E_j^C\} \\
|010\rangle_{ABC}: &\quad \{|E_i^A\rangle \otimes |E_j^B\rangle \otimes |E_i^C\rangle, \quad E_i^A + E_j^B + E_i^C\} \\
|011\rangle_{ABC}: &\quad \{|E_i^A\rangle \otimes |E_j^B\rangle \otimes |E_j^C\rangle, \quad E_i^A + E_j^B + E_j^C\} \\
|100\rangle_{ABC}: &\quad \{|E_j^A\rangle \otimes |E_i^B\rangle \otimes |E_i^C\rangle, \quad E_j^A + E_i^B + E_i^C\} \\
|101\rangle_{ABC}: &\quad \{|E_j^A\rangle \otimes |E_i^B\rangle \otimes |E_j^C\rangle, \quad E_j^A + E_i^B + E_j^C\} \\
|110\rangle_{ABC}: &\quad \{|E_j^A\rangle \otimes |E_j^B\rangle \otimes |E_i^C\rangle, \quad E_j^A + E_j^B + E_i^C\} \\
|111\rangle_{ABC}: &\quad \{|E_j^A\rangle \otimes |E_j^B\rangle \otimes |E_j^C\rangle, \quad E_j^A + E_j^B + E_j^C\}
\end{aligned}
$$

New virtual qubits arise between levels which cannot be tensored into virtual qubits of A, B, C, or any qubit within AB, BC or AC. this means that the two states that the qubit is constructed of have to be different in all entries. For example, $|000\rangle_{ABC}$ can paired with $|111\rangle_{ABC}$. All in all, the set of states can be divided into such pairs uniquely, with a total number of new virtual qubits $2^3/2 = 4$. Here we list them all:

- between $|000\rangle_{ABC}$ and $|111\rangle_{ABC}$:

$$
\begin{aligned}
e^{-\beta_v^1(\Delta E^A + \Delta E^B + \Delta E^C)} = \frac{p_j^A p_j^B p_j^C}{p_i^A p_i^B p_i^C} &= e^{-\beta_v^A \Delta E^A} e^{-\beta_v^B \Delta E^B} e^{-\beta_v^C \Delta E^C} \\
\beta_v^1 &= \beta_v^A \left(\frac{\Delta E^A}{\Delta E^A + \Delta E^B + \Delta E^C} \right) + \beta_v^B \left(\frac{\Delta E^B}{\Delta E^A + \Delta E^B + \Delta E^C} \right) \\
&\quad + \beta_v^C \left(\frac{\Delta E^C}{\Delta E^A + \Delta E^B + \Delta E^C} \right);
\end{aligned}
$$

- between $|001\rangle_{ABC}$ and $|110\rangle_{ABC}$:

$$
\begin{aligned}
e^{-\beta_v^2(\Delta E^A + \Delta E^B - \Delta E^C)} = \frac{p_j^A p_j^B p_i^C}{p_i^A p_i^B p_j^C} &= e^{-\beta_v^A \Delta E^A} e^{-\beta_v^B \Delta E^B} e^{+\beta_v^C \Delta E^C} \\
\beta_v^2 &= \beta_v^A \left(\frac{\Delta E^A}{\Delta E^A + \Delta E^B + \Delta E^C} \right) + \beta_v^B \left(\frac{\Delta E^B}{\Delta E^A + \Delta E^B + \Delta E^C} \right) \\
&\quad - \beta_v^C \left(\frac{\Delta E^C}{\Delta E^A + \Delta E^B + \Delta E^C} \right);
\end{aligned}
$$

- between $|011\rangle_{ABC}$ and $|100\rangle_{ABC}$:

$$
\begin{aligned}
e^{-\beta_v^3(\Delta E^A - \Delta E^B - \Delta E^C)} = \frac{p_j^A p_i^B p_i^C}{p_i^A p_j^B p_j^C} &= e^{-\beta_v^A \Delta E^A} e^{+\beta_v^B \Delta E^B} e^{+\beta_v^C \Delta E^C} \\
\beta_v^3 &= \beta_v^A \left(\frac{\Delta E^A}{\Delta E^A + \Delta E^B + \Delta E^C} \right) - \beta_v^B \left(\frac{\Delta E^B}{\Delta E^A + \Delta E^B + \Delta E^C} \right) \\
&\quad - \beta_v^C \left(\frac{\Delta E^C}{\Delta E^A + \Delta E^B + \Delta E^C} \right);
\end{aligned}
$$

- between $|010\rangle_{ABC}$ and $|101\rangle_{ABC}$:

$$e^{-\beta_v^4(\Delta E^A-\Delta E^B+\Delta E^C)} = \frac{p_j^A p_i^B p_j^C}{p_i^A p_j^B p_i^C} = e^{-\beta_v^A \Delta E^A} e^{+\beta_v^B \Delta E^B} e^{-\beta_v^C \Delta E^C}$$

$$\beta_v^4 = \beta_v^A \left(\frac{\Delta E^A}{\Delta E^A + \Delta E^B + \Delta E^C}\right) - \beta_v^B \left(\frac{\Delta E^B}{\Delta E^A + \Delta E^B + \Delta E^C}\right) + \beta_v^C \left(\frac{\Delta E^C}{\Delta E^A + \Delta E^B + \Delta E^C}\right);$$

(c) How many (new) virtual qubits can we extract from a composition of n virtual qubits?

Solution Following the logic of (b), the number of new virtual qubits is half of the dimension of the subspace: 2^{n-1}.

(d) Now take m copies of the virtual qubit we chose in A and n copies of the virtual qubit in B and calculate the virtual temperature of the pair $|0\rangle_A^{\otimes m} \otimes |1\rangle_B^{\otimes n}$ and $|1\rangle_A^{\otimes m} \otimes |0\rangle_B^{\otimes n}$.

Solution The energy gap between the indicated states is $\Delta E = m\Delta E^A - n\Delta E^B$. The virtual temperature is defined by

$$e^{-\beta_v(m\Delta E^A - n\Delta E^B)} = \frac{(p_j^A)^m (p_i^B)^n}{(p_i^A)^m (p_j^B)^n} = e^{-m\beta_v^A \Delta E^A} e^{+n\beta_v^B \Delta E^B}$$

$$\beta_v = \beta_v^A \left(\frac{m\Delta E^A}{m\Delta E^A + n\Delta E^B}\right) - \beta_v^B \left(\frac{n\Delta E^B}{m\Delta E^A + n\Delta E^B}\right).$$