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Advanced Topics in QIT
Solution 4

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Exercise 1. *Majorization: examples and properties*

(a) *(triangles) Let $\theta_1, \theta_2, \theta_3$ be the angles of a triangle, expressed in radians. Show that*

$$(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}) \prec (\theta_1, \theta_2, \theta_3) \prec (\pi, 0, 0).$$

Solution $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ is the uniform distribution over the angles, and hence majorized by everything else. As for the rest,

$$\theta_1^{\downarrow} \leq \theta_1^{\downarrow} + \theta_2^{\leq}\theta_1^{\downarrow} + \theta_2^{+}\theta_3^{\downarrow} = \pi,$$

which proves $(\theta_1, \theta_2, \theta_3) \prec (\pi, 0, 0)$.

(b) *(measurement outcomes) Let ρ be an arbitrary state of a d-dimensional quantum system. Prove that there always exists an orthonormal basis $|e_k\rangle$ such that the probabilities for a measurement in that basis are uniformly distributed. Given ρ, can you explicitly construct a basis $|e_k\rangle$ such that this is true?*

Hint: Use Schur-Horn theorem.

Solution Suppose a quantum system with d-dimensional state space is in a state described by a density matrix ρ with vector of eigenvalues $\lambda(\rho) = (\lambda_1(\rho), \ldots, \lambda_d(\rho))$, so that

$$\rho = \sum_j \lambda_j |j\rangle\langle j|.$$

Suppose that we measure it in an orthonormal basis $\{|e_k\rangle\}$, then the probability distribution over outcomes

$$p(k) = \sum_j \lambda_j \langle e_k|j\rangle\langle j|e_k\rangle = \sum_j \lambda_j |u_{jk}|^2, \text{ where} u_{jk} \text{ are entries of a unitary matrix.}$$

Then, according to the Schur-Horn theorem (or, more directly, Horn's lemma) $p(k)$ is majorized by $\lambda(\rho)$. Moreover, the converse is also true: if $p(k)$ is majorized by $\lambda(\rho)$, then there exists a corresponding measurement basis $\{|e_k\rangle\}$. Since the uniform distribution is majorized by everything else, so there indeed exists a basis $\{|e_k\rangle\}$ such that he probabilities for a measurement in that basis are uniformly distributed. Explicitly, this basis is a Fourier transform of the one where ρ is diagonal:

$$|e_k\rangle = \frac{1}{\sqrt{d}} \sum_j e^{\frac{2\pi i}{d} jk} |j\rangle, \ \langle e_k|j\rangle = \frac{1}{\sqrt{d}} e^{\frac{2\pi i}{d} jk}.$$

Then

$$p(k) = \sum_j \lambda_j \frac{1}{d} = \frac{1}{d}.$$

(c) *(post-measurement state) Let ρ be a density matrix, and $\{P_j\}$ a complete set of orthonormal projectors. Then the posterior density matrix*

$$\rho' = \sum_j P_j \rho P_j$$

satisfies $\rho' \prec \rho$.

Solution Given the set of projectors, let us define the following set of unitaries:

$$U_k = \sum_j e^{\frac{2\pi i}{d} jk} P_j.$$

Then the uniform mixture of these unitaries gives us the desired post-measurement state:

$$\sum_k \frac{1}{d} U_k \rho U_k^\dagger = \frac{1}{d} \sum_{k,j,l} e^{\frac{2\pi i}{d} jk} P_j \rho e^{-\frac{2\pi i}{d} lk} P_l = \sum_{j,l} \frac{1}{d} \sum_k e^{\frac{2\pi i}{d}(j-l)k} P_j \rho P_l = \sum_{j,l} \delta_{j,l} P_j \rho P_l = \sum_j P_j \rho P_j = \rho'.$$

Due to Uhlmann's theorem, this is equivalent to $\rho' \prec \rho$.

(d) *(doubly stochastic matrices) Show that if D and E are doubly stochastic, then DE is also doubly stochastic. Find an example of vectors r and y and a doubly stochastic matrix D such that $r \prec y$ but $Dr \not\prec Dy$.*

Solution All entries of DE are non-negative, as the entries of D and E are non-negative. Summing the rows and the columns, we get

$$\sum_i (DE)_{ij} = \sum_{i,k} D_{ik} E_{kj} = \sum_k \left(\sum_i D_{ik} \right) E_{kj} = \sum_k E_{kj} = 1$$

$$\sum_j (DE)_{ij} = \sum_{j,k} D_{ik} E_{kj} = \sum_k \left(\sum_i E_{kj} \right) D_{ik} = \sum_k D_{ik} = 1.$$

(e) *(thermal states) Compare two thermal states $\tau(\beta_1)$ and $\tau(\beta_2)$ with $\beta_1 < \beta_2$. Which state majorizes the other?*

Solution Starting with the ground state, the populations of energy levels decrease exponentially. For the state with the bigger inverse temperature β_2, the population of a given level is always higher than the population of the same level of the state with lower inverse temperature β_1. Hence, due to the majorization criterion,

$$\tau(\beta_1) \prec \tau(\beta_2).$$

(f) *(energy) Show that if $\rho_1 \succ \rho_2$, and ρ_1 and ρ_2 are passive, then ρ_1 has lower average energy: $E_1 < E_2$.*

Solution The average energy of a passive state, the eigenvalues of which are ordered non-increasingly with respect to energy eigenvalues increasing, is a Schur-concave function:

$$(p_i - p_j)(E_i - E_j) \leq 0.$$

Hence, $\rho_1 \succ \rho_2 \Rightarrow E_1 < E_2$.

Exercise 2. ***Yet again: optimal cooling***

Suppose that we are given a (real) qubit S with Hamiltonian $H_S = E_S|1\rangle\langle 1|_S$. Suppose that we also have access to a machine B – an d_B-dimensional system with the Hamiltonian $H_B = \sum_{i=0}^{d_B-1} \mathcal{E}_i |\mathcal{E}_i\rangle\langle\mathcal{E}_i|_B$, where the eigenvalues are ordered: $\mathcal{E}_i \leq \mathcal{E}_j$ for $i < j$; we denote $\mathcal{E}_{d_B-1} = \mathcal{E}_{max}$. The initial states of the machine and the qubit are thermal ones at the inverse temperature β: $\rho_S = \tau_S[\beta]$, $\rho_B = \tau_B[\beta]$. Assume that $\mathcal{E}_{max} > E_S$, and $\beta > 0$.

As before, we want to make the qubit S cooler (no pun intended).

(a) *Recall the setting from the previous exercise sheet. There, we have repeatedly performed a swap operation with one of the virtual qubits of the system B, resetting B after each round. Now apply the reasoning to this exercise. Swapping with which virtual qubit of B would lead to the real qubit having the lowest temperature possible, in the limit of the number of operations applied $n \to \infty$? What is this temperature β^*?*
Hint: $\beta^* = \frac{\beta\mathcal{E}_{max}}{E_S}$.

Solution The basic idea of this cooling procedure is to repeatedly swap with the virtual qubit which has the maximal energy gap, and, hence, the lowest virtual temperature (or the largest inverse temperature). Then, in the limit of infinite number of swaps, the final state of the cooled qubit will reach the temperature of the virtual one (for the derivation of this, refer to the previous exercise sheet or lecture notes). The virtual temperature of that maximal energy gap qubit, which uses states $|0\rangle_B$ and $|\mathcal{E}_{max}\rangle$, is

$$\beta^* = \frac{\mathcal{E}_{max}\beta}{E_S} > \beta > 0,$$

which gives us the desired bound. This bound is indeed the cooling bound; however, the procedure is not optimal.

(b) *The coolest achievable state of the qubit is $\rho_S^* = \tau_S[\beta^*]$. Show that it majorizes the initial state of the qubit.*

Solution The initial state of the qubit is $\tau_S[\beta]$, with $\beta < \beta^*$. The thermal state with a lower temperature always majorizes the state with the higher one: $\tau_S[\beta^*] \succ \tau_S[\beta]$.

Moreover, one can show that the following holds: for any unitary U, and any single coherent operation C defined as

$$C(\rho_S) = \mathrm{tr}_B(\ U(\rho_S \otimes \tau_B[\beta])U^\dagger\ ),$$

$C(\rho_S) \prec \rho_S^*$ *(you don't need to prove this here, just convince yourselves that it holds :)).*

But back to cooling: even though the operation in (a) achieves the cooling bound, it is, in fact, not optimal. Let us consider another operation instead: given a state ρ_{SB}, let U_{opt} be the unitary that reorders the eigenvalues of ρ_{SB} as largest in the energy subspace $|0\mathcal{E}_0\rangle\langle 0\mathcal{E}_0|_{SB}$, second largest in $|0\mathcal{E}_1\rangle\langle 0\mathcal{E}_1|_{SB}$, and so on up to $|1\mathcal{E}_{max}\rangle\langle 1\mathcal{E}_{max}|_{SB}$:

$$U_{opt}\ \rho_{SB}\ U_{opt}^\dagger = \sum_{i=0}^{d_B-1} \left([\rho_{SB}]_i\ |0\mathcal{E}_i\rangle\langle 0\mathcal{E}_i|_{SB}\ +\ [\rho_{SB}]_{d_B+i}\ |1\mathcal{E}_i\rangle\langle 1\mathcal{E}_i|_{SB}\right),$$

where $[\rho_{SB}]_i$ are entries of the vector of the diagonal entries of ρ_{SB}, arranged in non-increasing order.

Now we are interested in the operation $A(\rho_S)$:

$$A(\rho_S) = \mathrm{tr}_B(\ U_{opt}(\rho_S \otimes \tau_B[\beta])U_{opt}^\dagger\ ).$$

(c) ***(*)** Show that A is optimal: for any C, $A^{\cdot n}(\rho_S) \succ C^{\cdot n}(\rho_S)$.*

Hint: Use that if $\rho_1 \succ \rho_2$, then $\rho_1 \otimes \tau_B[\beta] \succ \rho_2 \otimes \tau_B[\beta]$. By induction, follow the steps:

1. *prove the statement for $n = 1$: $A(\rho_S) \succ C(\rho_S)$;*
2. *prove $\rho_1 \succ \rho_2 \Rightarrow A(\rho_1) \succ C(\rho_2)$;*
3. *assume $A^{\cdot n}(\rho_S) \succ C^{\cdot n}(\rho_S)$, and show $A^{\cdot(n+1)}(\rho_S) \succ C^{\cdot(n+1)}(\rho_S)$.*

Solution We prove the statement by induction. We start with the base case $n = 1$, i.e. a single coherent operation. From the argument in section A the largest possible value of the partial sum of the k largest eigenvalues of the final state ρ'_S is the sum of the $k * d_B$ largest eigenvalues of the joint state ρ_{SM}.

Consider a single application of the optimal coherent protocol. Tracing out the machine to find the final state of the system, it is diagonal in the energy eigenbasis,

$$\rho'_S = \left(\sum_{j=0}^{d_B-1} [\vec{\rho}_{SB}]_j |0\rangle\langle 0| + [\vec{\rho}_{SB}]_{d_B+j} |1\rangle\langle 1| \right).$$

Since $\vec{\rho}_{SB}$ is the vector of eigenvalues of the joint state arranged in decreasing order, it follows that the ground state population of the final state of the system, i.e. $\langle 0|\rho'_S|0\rangle$, is composed of the sum of the largest d_B eigenvalues of ρ_{SB}, and the first excited state is the sum of the next largest eigenvalues of ρ_{SB}, and so on. Thus the partial sum of the largest k eigenvalues of ρ'_S is the sum of the $k * d_B$ largest eigenvalues of the joint state. This concludes the proof for $n = 1$.

Next we prove that if $\rho_1 \succ \rho_2$, then $A(\rho_1) \succ C(\rho_2)$, for an arbitrary coherent operation C. Firstly, if $\rho_1 \succ \rho_2$, then it is also true that $\rho_1 \otimes \tau_B(\beta) \succ \rho_2 \otimes \tau_B(\beta)$. But then the sum of the largest $k * d_B$ eigenvalues of $\rho_1 \otimes \tau_B(\beta)$ is larger than (or equal to) that in the second case. Thus $A(\rho_1) \succ A(\rho_2)$. But we have proven above that $A(\rho_2) \succ C(\rho_2)$. Therefore $A(\rho_1) \succ C(\rho_2)$.

Finally, assume that $A^{\otimes n}(\rho_S) \succ C^{\otimes n}(\rho_S)$. Let $\rho_1 = A^{\otimes n}(\rho_S)$ and $\rho_2 = C^{\otimes n}(\rho_S)$. Then from the previous argument $A^{\otimes(n+1)}(\rho_S) \succ C^{\otimes(n+1)}(\rho_S)$, which concludes the proof.

(d) ***(*)** Show that $A^\infty(\rho_S) \lim_{n\to\infty} A^n(\rho_S) = \rho_S^*$. Thus, we conclude that A is the optimal (fastest) operation which achieves the cooling bound.*

Solution Here we can finally prove that the optimal coherent protocol also converges to the desired state. Any cooled state of the qubit is majorized by the state given by the cooling bound:

$$\sum_{i=0}^{k-1} [A^\infty(\rho_S)]_i \leq \sum_{i=0}^{k-1} [\rho_S^*]_i, \quad \text{for } k = 1, 2.$$

From the previous consideration we have that application of A majorizes the application of B:

$$\sum_{i=0}^{k-1} [B^n(\rho_S)]_i \leq \sum_{i=0}^{k-1} [A^n(\rho_S)]_i, \quad \text{for } k = 1, 2, \forall n \in \mathbb{N}.$$

And so

$$\sum_{i=0}^{k-1} [\rho_S^*]_i = \sum_{i=0}^{k-1} [B^\infty(\rho_S)]_i \leq \sum_{i=0}^{k-1} [A^\infty(\rho_S)]_i = \sum_{i=0}^{k-1} [\rho_S^*]_i, \quad \text{for } k = 1, 2.$$

Therefore for all $k = 1, 2$ we have $\sum_{i=0}^{k-1}[A^\infty(\rho_S)]_i = \sum_{i=0}^{k-1}[\rho_S^*]$ and

$$\begin{aligned} A^\infty(\rho_S) &= \sum_{k=0}^{1} \left(\sum_{i=0}^{k+1}[A^\infty(\rho_S)]_i - \sum_{j=0}^{k}[A^\infty(\rho_S)]_j \right) |k\rangle\langle k| \\ &= \sum_{k=0}^{1} \left(\sum_{i=0}^{k+1}[\rho_S^*]_i - \sum_{j=0}^{k}[\rho_S^*]_j \right) |k\rangle\langle k| \\ &= \rho_S^*. \end{aligned}$$