Exercise 1. *Lorenz curves*

In this exercise, we will see that Lorenz curves is a useful tool for illustrating the majorization concept for different states.

(a) *Consider a six dimensional system which composed of a qubit S and a qutrit R. We will look at two states: the first state ρ_1 is a composition of a pure state of a qubit $|\psi\rangle_S$ and the maximally mixed state of a qutrit*

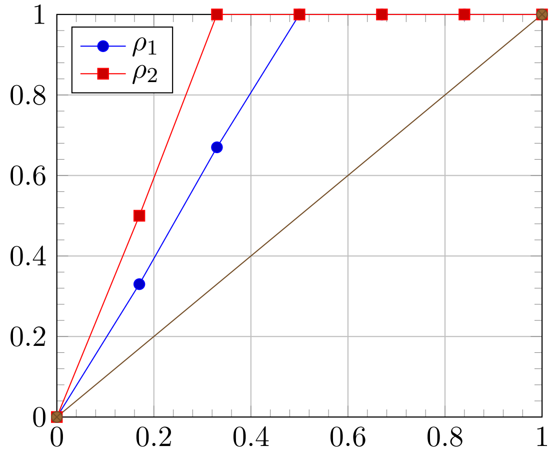
$$\rho_1 = |\psi\rangle\langle\psi|_S \otimes \frac{1}{3}\mathbb{1}_R;$$

and the second state ρ_2 is a composition of a pure state of a qutrit $|\phi\rangle_R$ and the maximally mixed state of the qubit

$$\rho_2 = \frac{1}{2}\mathbb{1}_S \otimes |\phi\rangle\langle\phi|_R.$$

Draw Lorenz curves for both states. What is the difference?

Solution The Lorenz curves for both states are portrayed below:



The plot corresponding to ρ_2 lies higher – which means that the state is less mixed, and more resource can be extracted from it.

(b) *Lorenz curves are useful not only in quantum thermodynamics; moreover, they were invented for a completely different purpose. Assume that the probability distribution is the income distribution: the percentage of GDP (gross domestic product) that constitutes the income of a member of the population. Which curves correspond to perfect (economic) equality and perfect inequality?*

Hint: Not a hint, but a suggestion: search for the Lorenz curves of different countries. This is surprisingly a very illustrative way to assess how economically equal the society is!

Solution Perfect equality corresponds to the straight line (a maximally mixed state, a uniform distribution). Perfect inequality corresponds to the plot of a pure state.

Exercise 2. ***Sets of noisy operations***

In this exercise, we investigate the structure of two sets: the set of noisy classical operations, and the set of noisy quantum operations. For completeness, we also list the definitions of all operations mentioned.

Definition 1 (Noisy classical operation)**.** *A noisy classical operation is a positivity-preserving and normalization-preserving map $D : V_{in} \to V_{out}$ that admits the following decomposition: there exists an ancilla system with a discrete physical space Ω_a and a permutation on $\Omega_{in} \times \Omega_a$ with an induced representation π on $V(\Omega_{in}) \otimes V(\Omega_a)$ such that for all input states x_{in}*

$$Dx_{in} = \sum_{\Omega_{a'}} \pi(x_{in} \otimes m_a),$$

where m_a is the normalized uniform distribution on Ω_a, and $\Omega_{a'}$ is the physical state space complementary to Ω_{out}: $\Omega_{in} \times \Omega_a = \Omega_{out} \times \Omega_{a'}$.

Definition 2 (Uniform-preserving stochastic map)**.** *A stochastic map D is uniform-preserving if it takes a uniform distribution on the input system to the uniform distribution on the output system.*

(a) What are the properties of the uniform-preserving stochastic matrices? What can you say about the case when the input and output spaces are of equal dimension $d_{in} = d_{out}$?

Solution Due to the definition, (D_{ij}) has non-negative entries, its columns sum up to 1: $\sum_i D_{ij} = 1$, and its columns sum up to $\frac{d_{in}}{d_{out}}$: $\sum_j D_{ij} = \frac{d_{in}}{d_{out}}$, where $d_{in}(d_{out})$ is the dimension of the input (output) vector space. When the input and output spaces are of equal dimension $d_{in} = d_{out}$, D is doubly stochastic.

(b) Prove that the set of noisy classical operations coincides with the set of uniform-preserving stochastic matrices and, in the case of equal dimension of input and output spaces, it coincides with the set of mixtures of permutations.

Solution It is straightforward to see that noisy classical operations are stochastic matrices (per definition). Moreover, if the input distribution is uniform, then by adjoining an ancilla in a uniform state and implementing a permutation, one creates a uniform state on the whole space, and every marginal is then also a uniform state.

The converse direction, that every uniform-preserving stochastic matrix can be realized as a noisy operation, is the nontrivial one. When the input and output vector spaces are of equal dimension, the stochastic uniform-preserving matrices are doubly stochastic, and a famous result due to Birkhoff establishes that every doubly stochastic matrix is achievable as a mixture of permutations. Therefore, it suffices to show that every mixture of permutations can be realized as a noisy classical operation. To implement permutation P_i with probability p_i, prepare an arbitrarily large ancilla in the uniform state, and partition its sample space into subsets of relative size p_i. Next, implement a controlled permutation with the ancilla as the control and the system as the target, where, if the ancilla is found in the subset i, the permutation P_i is implemented on the system. Such a controlled permutation is, of course, itself a permutation on the composite of system and ancilla. Finally, discard the ancilla.

Finally, we can show that even if the input and output spaces have different dimensions, every uniform-preserving stochastic matrix can be achieved as a noisy classical operation.

Let D be a uniform-preserving stochastic matrix from a d_{in}-dimensional probability space to a d_{out}-dimensional one. We now append an ancillary system to the input and one to the output such that the two composites are of equal dimension. That is, we define ancillary systems of dimensions d_1 and d_2 such that $d_{in}d_1 = d_{out}d_2$. Next, we define D_0 to be the $d_2 \times d_1$ matrix all of whose entries are $1/d_2$. This is clearly just the stochastic matrix that maps every state of dimension d_1 to the uniform state of dimension d_2 and is consequently uniform-preserving. It follows that the $(d_{out}d_2) \times (d_{in}d_1)$ matrix $D \otimes D_0$ is also a uniform-preserving stochastic matrix. Given that $d_{out}d_2 = d_{in}d_1$, it follows that $D \otimes D_0$ is doubly-stochastic and hence can be implemented by a mixture of permutations.

It follows that, if D is any uniform-preserving stochastic matrix, then it can be implemented by first adjoining a uniform state of dimension d_1 (which is a noisy operation), implementing a mixture of permutations (which, as shown earlier in this proof, is a noisy operation), and finally marginalizing over the ancillary subsystem of dimension d_2 (which is also a noisy operation). Given that every step of the implementation is a noisy operation, the overall operation is, as well.

Now let us turn to the noisy quantum operations.

Definition 3 (Noisy quantum operation)**.** *A noisy quantum operation $\mathcal{E}$ is one that admits the following decomposition: there exist a finite-dimensional ancilla space $\mathcal{H}_a$ and a unitary U on $\mathcal{H}_{in} \otimes \mathcal{H}_a$ such that for all input states ρ_{in}*

$$\mathcal{E}(\rho_{in}) = \mathrm{tr}_{a'}\left(U\left(\rho_{in} \otimes \frac{1}{d_a}\mathbb{1}_a\right)U^\dagger\right),$$

where $\mathcal{H}_{\dashv'}$ is complementary to $\mathcal{H}_{out}$ in the total Hilbert space: $\mathcal{H}_{in} \otimes \mathcal{H}_a = \mathcal{H}_{out} \otimes \mathcal{H}_{a'}$, and $d_a = \dim \mathcal{H}_a$.

A unital operation is the quantum analog of a uniform-preserving stochastic map:

Definition 4 (Unital operations)**.** *An operation $\mathcal{E}$ is unital if it maps a maximally mixed state to a maximally mixed state:*

$$\mathcal{E}(\frac{1}{d_{in}}\mathbb{1}_{in}) = \frac{1}{d_{out}}\mathbb{1}_{out}.$$

However, the relationship between the set of unital operations and the set of quantum noisy operations is not as straightforward as for their classical counterparts. Noisy quantum operations form a strict subset of the unital operations, and, in the case of equal dimension of input and output space, a strict superset of the mixtures of unitaries.

(c) *Prove that a noisy quantum operation is necessarily unital.*

Solution This is similar to the classical case. Adding an ancilla in a maximally mixed keeps the overall state maximally mixed, as well as applying a unitary and consequently marginalizing it. Hence, we get a maximally mixed state as an output.

(d) *Prove that a mixture of unitaries is necessarily a noisy quantum operation.*
Hint: Given an ensemble of unitaries (p_i, U_i), consider an ancilla (of an arbitrarily large dimension) in the completely mixed state, and partition its Hilbert space into subspaces, the relative dimensions of which are described by the distribution $\{p_i\}$.

Solution The procedure here is very similar to the classical case considered before. Let us introduce an ancilla of dimension d one implements a controlled unitary with the ancilla as the control and the system as the target, where, if the ancilla is found in the subspace i, the unitary U_i is implemented on the system. Such a controlled unitary is, of course, itself a unitary on the composite of system and ancilla. Finally, one discards the ancilla – and arrives to the desired action of the mix of unitaries.

Proving strictness of the inclusion is more complicated: example of a unital operation that is not a quantum noisy operation is provided in (Haagerup and Musat, 2011), and the fact that not every quantum noisy operation is a mixture of unitaries has been shown by Shor in (Shor, 2010).

References:

Haagerup, U., and M. Musat, 2011, Commun. Math. Phys. 303(2), 555.

Shor, P. W., 2010, Structure of Unital Maps and the Asymptotic Quantum Birkhoff Conjecture, presentation.

Exercise 3. ***Thermal bath heat capacity***

Heat capacity of a system can be defined as the rate of change of inverse temperature with adding heat, $\kappa = d\beta/dQ$.

(a) *Calculate the heat capacity for a single system in a thermal state* $\tau[\beta]$.

Solution

$$\kappa = \frac{d\beta}{dQ} = \left(\frac{dE}{d\beta}\right)^{-1} = \left(\frac{-Z\sum_i E_i^2 e^{-\beta E_i} + \sum_i E_i e^{-\beta E_i} \cdot \sum_i E_i e^{-\beta E_i}}{Z^2}\right)^{-1} = \frac{1}{\langle E\rangle^2 - \langle E^2\rangle}.$$

One way to arrive to the thermal bath is to take n *copies of a system in a thermal state:*

$$\rho = \tau[\beta]^{\otimes n}.$$

(b) *Show that the heat capacity of this composite system scales as* n^{-1}.

Solution The heat capacity for n systems (since they are independent and identical) can be written as

$$\kappa_n = \left(\frac{dE_n}{d\beta}\right)^{-1} = \left(\sum_{j=1}^{n} \frac{dE}{d\beta}\right)^{-1} = \left(\sum_{j=1}^{n} \left(\langle E\rangle^2 - \langle E^2\rangle\right)\right)^{-1} = \frac{\kappa}{n}.$$