Exercise 1. *Lindbladian evolution*

Lindbladian evolution is an extension of Schroedinger's equation to open systems. In this exercise, we will walk through necessary steps to derive it.

(a) *Write down the evolution of a state $\rho(t)$ with Hamiltonian H over a tiny period of time δt. Use Taylor expansion for the exponential, and express $\rho(t+\delta t)$ in terms of $\rho(t)$ up to the first order in δt. Take the limit $\delta t \to 0$. This is the time-independent Schrödinger equation for mixed states.*

Solution The Taylor expansion of $U(\delta t)$:

$$U(\delta t) = \exp\left(-\frac{i\ \delta t}{\hbar} H\right) = \mathbb{1} - \frac{i\ \delta t}{\hbar} H + (\delta t^2).$$

Using this expansion, let us rewrite the expression above:

$$\begin{aligned}
\rho(t+\delta t) &= \left(\mathbb{1} - \frac{i\ \delta t}{\hbar} H + (\delta t^2)\right) \rho(t) \left(\mathbb{1} + \frac{i\ \delta t}{\hbar} H + (\delta t^2)\right) \\
&= \rho(t) + \frac{i\ \delta t}{\hbar} \mathbb{1}\rho(t) H - \frac{i\ \delta t}{\hbar} H \rho(t) \mathbb{1} + (\delta t^2) \\
&= \rho(t) - \frac{i\ \delta t}{\hbar} H\rho(t) + \frac{i\ \delta t}{\hbar}\rho(t) H + (\delta t^2) \\
&= \rho(t) - \frac{i\ \delta t}{\hbar}[H, \rho(t)] + (\delta t^2)
\end{aligned}$$

By rearranging the terms we obtain:

$$\frac{\rho(t+\delta t) - \rho(t)}{\delta t} = -\frac{i}{\hbar}[H, \rho(t)] = \frac{1}{i\hbar}[H, \rho(t)]$$

and, by taking the limit for $\delta t \to 0$ we obtain exactly the derivative on the left-hand side:

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H, \rho] \tag{S.1}$$

and this is the time-independent Schrödinger equation for mixed states.

Next we consider the notion of open systems: the concept of "openness" in general physics implies that the system can interact with an external environment. In order to schematize this in quantum theory, we consider a Hilbert space of the form:

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

where S is our open system and E represents the environment. Generally, we assume that the environment is a much larger system: $\dim \mathcal{H}_S \ll \dim \mathcal{H}_E$. To obtain the dynamics of the joint system, we need to solve the Schrödinger equation:

$$H_{SE} \mapsto U_{SE}(t) = \exp\left(-\frac{it}{\hbar} H_{SE}\right)$$

However, we may not even know all the details of the interaction enough to approximate the Hamiltonian. On the other hand, we do not need to keep track of the evolution of the state ρ_{SE} of the whole system, in particular it suffices for us to know enough of the state of the system ρ_S, i.e. the partial trace

$$\rho_S(t) = \mathrm{tr}_E\left(U_{SE}(t)\ \rho_{SE}(0)\ U_{SE}^\dagger(t)\right)$$

We can also describe the operation as a TPCPM $\mathcal{E}$ acting on the system S alone, which approximates what happens in the environment:

$$\rho_S(t) = \mathcal{E}_S(t, \rho_S(0))$$

In order for this approximation to be good enough, we assume that the dynamics of the environment is much faster than the interaction, which means that previous correlations between the system and the environment become negligible.

(b) *Consider a variation $\delta\rho$, $|\delta\rho| \ll |\rho|$:*

$$\mathcal{E}(\delta t, \rho_S(t)) \simeq \rho(t) + \delta\rho$$

Write down the Kraus decomposition of the channel. Let us rewrite them in the following way:

$$A_0 = \mathbb{1} + \delta t(L_0 - iK)$$
$$A_k = L_k\sqrt{\delta t},\ k > 0$$

where K and L_k for every k are bounded operators. Analyze the terms of the sum in the Kraus decomposition.

Solution For $k = 0$

$$\begin{aligned} A_0\rho A_0^\dagger &= (\mathbb{1} + \delta t\ (L_0 - iK))\ \rho\ (\mathbb{1} + \delta t\ (L_0 + iK)) \\ &= \rho + \delta t L_0\rho - i\delta t K\rho + \delta t\rho L_0 + i\delta t\rho K + (\delta t^2) \\ &= \rho + \delta t\{L_0, \rho\} + i\delta t[\rho, K] + (\delta t^2); \end{aligned}$$

the terms with $k > 0$

$$A_k\rho A_k^\dagger = L_k\rho L_k^\dagger\ \delta t.$$

(c) *Letting $\delta t \to 0$, show that*

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H_S, \rho] + \{L_0, \rho\} + \sum_{k=1}^{\infty} L_k\rho L_k^\dagger.$$

Hint: Choose $K = H_S/\hbar$, using the intuition from (a).

Solution If we put everything together we obtain:

$$\begin{aligned} \rho(t) + \delta\rho = \sum_{k=0}^{\infty} A_k\rho A_k^\dagger \\ &= \rho + \delta t\{L_0, \rho\} + i\delta t[\rho, K] + \delta t\sum_{k=1}^{\infty} L_k\rho L_k^\dagger + (\delta t^2). \end{aligned}$$

Hence, remembering that $\rho + \delta\rho = \rho(t + \delta t)$ and letting $\delta t \to 0$, we are left with:

$$\begin{aligned} \rho(t + \delta t) &= \rho(t) + \delta t\left(\{L_0, \rho\} + i[\rho, K] + \sum_{k=1}^{\infty} L_k\rho L_k^\dagger\right) \\ \frac{d\rho}{dt} &= \{L_0, \rho\} - i[K, \rho] + \sum_{k=1}^{\infty} L_k\rho L_k^\dagger. \end{aligned}$$

The term in the middle closely resembles what we obtained in the Schrödinger equation in (a), thus let us pick $K = H_S/\hbar$. The Lindblad equation now becomes:

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H_S, \rho] + \{L_0, \rho\} + \sum_{k=1}^{\infty} L_k \rho L_k^\dagger.$$

(d) *Suppose that S and E don't interact. Determine L_k in this case.*

Solution Let us consider an isolated system, i.e. the system S and the environment E do not interact:

$$H_{SE} = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E \Longrightarrow U_{SE}(t) = U_S(t) \otimes U_E(t)$$

implying that the evolution of our state is:

$$U_S(t)\rho_S(0)U_S^\dagger(t)$$

which means that this state satisfies the Schrödinger equation and, in turn, the Lindblad equation with $L_k \equiv 0$ for every k.

(e) *Using that the trace of the density matrix is always preserved* $\operatorname{tr}\left(\frac{d\rho}{dt}\right) = 0$*, find the expression for L_0 in terms of all other L_k, $k \neq 0$.*

Solution Let us check that the trace is preserved:

$$\operatorname{tr}\rho \equiv 1 \Longrightarrow \frac{d}{dt}\operatorname{tr}\rho = \operatorname{tr}\left(\frac{d\rho}{dt}\right) = 0,$$

where the last equality follows from linearity of trace. Now we replace the Lindblad equation here, and apply all the properties of trace we know:

$$0 = \operatorname{tr}\left(\frac{d\rho}{dt}\right) = \frac{1}{i\hbar}\operatorname{tr}[H_S, \rho] + \operatorname{tr}\{L_0, \rho\} + \sum_{k=1}^{\infty}\operatorname{tr}\left(L_k \rho L_k^\dagger\right)$$

The trace of a commutator is always zero, and the cyclic property of trace also ensures that

$$\operatorname{tr}(L_0\rho + \rho L_0) = \operatorname{tr}(L_0\rho) + \operatorname{tr}(\rho L_0) = 2\operatorname{tr}(L_0\rho).$$

The sum in the third term, on the other hand, yields

$$\sum_{k=1}^{\infty}\operatorname{tr}\left(L_k\rho L_k^\dagger\right) = \sum_{k=1}^{\infty}\operatorname{tr}\left(L_k^\dagger L_k \rho\right)$$

Putting all together we obtain that:

$$0 = 2\operatorname{tr}(L_0\rho) + \sum_{k=1}^{\infty}\operatorname{tr}\left(L_k^\dagger L_k\rho\right)$$

$$0 = \operatorname{tr}(L_0\rho) + \frac{1}{2}\sum_{k=1}^{\infty}\operatorname{tr}\left(L_k^\dagger L_k\rho\right)$$

$$0 = \operatorname{tr}\left(L_0\rho + \frac{1}{2}\sum_{k=1}^{\infty}L_k^\dagger L_k\rho\right)$$

$$0 = \operatorname{tr}\left(\left(L_0 + \frac{1}{2}\sum_{k=1}^{\infty}L_k^\dagger L_k\right)\rho\right)$$

Since this must hold for any state ρ, we evince that the expression in the tuples must be the null operator, i.e.

$$L_0 + \frac{1}{2}\sum_{k=1}^{\infty} L_k^\dagger L_k = 0 \Longleftrightarrow L_0 = -\frac{1}{2}\sum_{k=1}^{\infty} L_k^\dagger L_k$$

Hence we can replace L_0 in the Lindblad equation:

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H_S, \rho] - \frac{1}{2}\left\{\sum_{k=1}^{\infty} L_k^\dagger L_k, \rho\right\} + \sum_{k=1}^{\infty} L_k \rho L_k^\dagger.$$

Exercise 2. ***Generating entanglement with a thermal machine [1]***

We consider two qubits with identical energy gaps E weakly coupled to each other and to separate thermal reservoirs. We denote the ground and excited states $|0\rangle$, $|1\rangle$, and the free Hamiltonian for the qubits in this basis is

$$\hat{H}_0 = E(|1\rangle\langle 1| \otimes \mathbb{1} + \mathbb{1} \otimes |1\rangle\langle 1|),$$

The interaction Hamiltonian, which is energy conserving, is given by

$$\hat{H}_{int} = g(|10\rangle\langle 01| + |01\rangle\langle 10|),$$

where g is the strength of the coupling between the qubits. The coupling to the thermal baths is modelled using a simple collision model, where thermalisation happens through rare but strong events. At every time step, each qubit k is either reset to a thermal state τ_k at the temperature of its bath with a small probability or left unchanged. The state of the qubits evolves according to the master equation

$$\frac{\partial \rho}{\partial t} = i[\rho, \hat{H}_0 + \hat{H}_{int}] + \sum_{k\in\{c,h\}} p_k(\Phi_k(\rho) - \rho)$$

where p_k is the thermalisation rate for qubit k and $\Phi_c(\rho) = \tau_c \otimes \mathrm{tr}_c\, \rho$ and $\Phi_h(\rho) = \mathrm{tr}_h\, \rho \otimes \tau_h$. We take the first qubit to have the colder and the second to have the warmer bath temperature. We refer to them as the 'cold' and 'hot' qubit respectively and use subscripts c and h. The thermal states are given by $\tau_k = r_k|0\rangle\langle 0| + (1 - r_k)|1\rangle\langle 1|$ with occupation probabilities determined by the Boltzmann factor according to $r_k = 1/(1 + e^{-E/T_k})$ where T_k is the reservoir temperature for qubit k (we set $k_B = 1$ and $\hbar = 1$).

Note: the master equation applies only in the perturbative regime $p_c, p_h, g \ll E$ and $p_c, p_h \ll 1$.

Here, we want to obtain the steady state solution of the master equation.

(a) Check that the following is a steady solution of the master equation:

$$\rho_\infty = \gamma\left[p_c p_h \tau_c \otimes \tau_h + \frac{2g^2}{(p_c + p_h)^2}(p_c\tau_c + p_h\tau_h)^{\otimes 2} + \frac{g p_c p_h (r_c - r_h)}{p_c + p_h}\mathcal{Y}\right]$$

with $\mathcal{Y} = i|01\rangle\langle 10| - i|10\rangle\langle 01|$ and $\gamma = 1/(2g^2 + p_c p_h)$, and where $\rho^{\otimes 2} = \rho \otimes \rho$. Note that for resonant qubits, the steady state depends on the energy E only through r_c, r_h.

Solution There is no particular trick. You can use Mathematica :)

(b) Let us now determine how entangled the steady state is. To quantify this, use the notion of concurrence [2], which is an entanglement monotone defined for a mixed state of two qubits as:

$$\mathcal{C}(\rho) \equiv \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues (in decreasing order) of the Hermitian matrix $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

Solution For the steady state the concurrence can be written as

$$C(\rho_\infty) = \max\left\{0, f(r_c, r_h) - \sqrt{h(r_c, r_h)h(1-r_c, 1-r_h)}\right\}$$

with

$$f(r_c, r_h) = \gamma\frac{gp_cp_h}{p_c+p_h}|r_c - r_h|,$$
$$h(r_c, r_h) = \gamma\left(p_cp_hr_cr_h + 2g^2\left(\frac{p_cr_c + p_hr_h}{p_c+p_h}\right)^2\right).$$

(c) Consider the cases with $T_c = T_h$, and $T_h \to \infty$ and $T_c = 0$.

Solution When the two temperatures coincide, i.e. $T_c = T_h$, we have $C(\rho_\infty) = 0$ since $f(r_c, r_h) = 0$ in this case. That is, at equilibrium the steady state of the two qubits is always separable. However, when moving away from equilibrium by choosing different temperatures for the two baths, hence establishing a heat current from the hot to the cold bath, steady-state entanglement can be generated.

For the case with $T_h \to \infty$ and $T_c = 0$, we have $r_c = 1$ and $r_h = \frac{1}{2}$, and

$$f(r_c, r_h) = \frac{\gamma gp_cp_h}{2(p_c+p_h)}$$
$$h(r_c, r_h) = \frac{\gamma p_cp_h}{2} + \frac{2\gamma g^2(p_c+p_h/2)^2}{(p_c+p_h)^2}$$
$$h(1-r_c, 1-r_h) = \frac{2g^2\gamma p_h^2}{4(p_c+p_h)^2}.$$

The concurrence

$$C(\rho_\infty) = \max\left\{0, f(r_c, r_h) - \sqrt{h(r_c, r_h)h(1-r_c, 1-r_h)}\right\}$$
$$= \max\left\{0, \frac{\gamma gp_h}{2(p_c+p_h)}\left(p_c - \sqrt{p_cp_h + 4g^2\left(\frac{p_c+p_h/2}{p_c+p_h}\right)^2}\right)\right\}.$$

For example, for $g \approx 1.6\times10^{-3}$, $p_c \approx 10^{-2}$, $p_h \approx 1.1\times10^{-3}$, we obtain $C(\rho_\infty) \approx 0.054$.

References

[1] Jonatan Bohr Brask, Nicolas Brunner, Géraldine Haack, Marcus Huber, Autonomous quantum thermal machine for generating steady-state entanglement. ***New Journal of Physics 17, 113029 (2015).*** *arXiv:1504.00187.*
[2] William K. Wootters, Entanglement of Formation of an Arbitrary State of Two Qubits. ***Physical Review Letters, 10(80) (1998).*** *DOI: 10.1103/physrevlett.80.2245.*