Exercise 1. ***The dynamics of the ideal clock***

Let us consider an idealized clock: a system C with the Hamiltonian $\hat{H} = \hat{p}_c$.

(a) *What is the distinguishable basis of the time states for such a Hamiltonian? Write down the natural evolution of the clock. What does a time translation physically correspond to?*

Solution Given that the Hamiltonian of the clock is $\hat{H} = \hat{p}_c$, the generalised eigenvectors of the position operator $|x\rangle$ are a distinguishable basis of time states ($\langle x|x'\rangle = \delta(x - x')$), and that given any initial generalized eigenvector $|x\rangle$, the natural evolution of the clock will past through all of the positions $x' > x$,

$$e^{-it\hat{H}}|x\rangle = |x + t\rangle,$$

i.e. a time translation is equivalent to a spatial translation.

(b) *Show that this clock is "continuous": the temporal and spatial translations are equivalent for any state of the clock.*

Solution Note that the equivalence between time and space translations hold for any state of the clock,

$$\langle x|e^{-it\hat{H}}|\Psi\rangle = \langle x - t|\Psi\rangle.$$

The fact that this statement holds for all $x, t \in \mathbb{R}$, and in particular, for arbitrarily small t is what we refer to as continuity, or by referring to the clock as continuous.

(c) *Let us add a position-dependent potential $V(x_c)$ (which happens to be an infinitely differentiable function of compact support on $L_2(\mathbb{R})$) to the clock. Show that the clock still remains continuous, only gaining a phase from the potential*

$$\langle x|e^{-it\left(\hat{H}+V(\hat{x}_c)\right)}|\psi_c\rangle = e^{-i\int_{x-t}^{x} V(x')dx'}\langle x - t|\psi_c\rangle, \quad x, t \in \mathbb{R}.$$

Solution The idealised clock remains continuous under the action of a potential:

$$\langle x|e^{-i(\hat{p}_c+V(\hat{x}_c))t}|\Psi\rangle = e^{-i\int_{x-t}^{x} V(x')dx'}\langle x - t|\Psi\rangle,$$

$V \in D_0$, $x, t \in \mathbb{R}$. Labelling the wavefunction $\langle x|e^{-i(\hat{p}_c+V(\hat{x}_c))t}|\Psi\rangle$ as $\psi(x, t)$, then the expression above is equivalent to

$$\psi(x, t) = \psi(x - t, 0)e^{-i\int_{x-t}^{x} V(x')dx'}. \tag{S.1}$$

We proceed to verify that it is the solution to Schrodinger's equation for the clock,

$$i\frac{\partial}{\partial t}\psi(x, t) = -i\frac{\partial}{\partial x}\psi(x, t) + V(x)\psi(x, t). \tag{S.2}$$

Direct calculation gives

$$\frac{\partial}{\partial t}\psi(x,t) = -\left(\frac{\partial}{\partial x}\psi(x-t,0)\right) e^{-i\int_{x-t}^{x} V(x')dx'} - iV(x-t)\psi(x,t), \tag{S.3}$$

$$\frac{\partial}{\partial x}\psi(x,t) = +\left(\frac{\partial}{\partial x}\psi(x-t,0)\right) e^{-i\int_{x-t}^{x} V(x')dx'} - i\left(V(x)-V(x-t)\right)\psi(x,t), \tag{S.4}$$

from which it is easily verified that $\psi(x,t)$ satisfies the Schrödinger equation. Hence, if one adds a position-dependent potential to the clock, it still remains continuous, while its state is only modified by a phase that depends on the potential. The phase integrates over the potential in the region that the state passes through ($[x-t,x]$). The clock can therefore be turned into a control device by simply having the potential be an interaction on an external system, whose strength is a function of the clock's position. Rather than an observer having to switch on and off an interaction on a system, here the clock does so autonomously by passing through the region of the potential.

Now take a step back, and consider a system S in a state $\rho_s \in \mathcal{S}(\mathcal{H}_s)$, upon which one wishes to perform the (energy-preserving) unitary U_s over a time interval $[t_i,t_f]$. The unitary can be implemented by the addition of $\hat{H}_s^{int}$ as a time-dependent interaction

$$\hat{H} = \hat{H}_s + \hat{H}_s^{int}\cdot V(t),$$

where $V(t)$ is a normalized pulse, i.e.

$$\int_{t_i}^{t_f} dt V(t) = 1.$$

To get rid of the explicit time-dependence of the Hamiltonian, we add a second system: a clock, and replace the background time parameter t by the time degree of the clock. For the idealised clock, this corresponds to a Hamiltonian on $\mathcal{H}=\mathcal{H}_s\otimes\mathcal{H}_c$, given by

$$\hat{H}_{total}^{id} = \hat{H}_s\otimes \mathbb{1}_c + \mathbb{1}_s \otimes \hat{p}_c + \hat{H}_s^{int}\otimes V(\hat{x}_c).$$

(d) *Take the initial state of the system and clock be in a product form*

$$|\psi_{sc}^0\rangle = |\psi_s^0\rangle \otimes |\psi_c^0\rangle,$$

and assume that $H_s=0$. Verify that this Hamiltonian can indeed implement the unitary U_s.

Solution Given the initial system state $\rho_s\in\mathcal{S}(\mathcal{H}_s)$, the state at the time t is

$$\rho_s(t) = e^{-it\hat{H}_s} e^{-i\hat{H}_s^{int}\int_{t_i}^{t} dx\, V(x)} \rho_s\, e^{+i\hat{H}_s^{int}\int_{t_i}^{t} dx\, V(x)} e^{+it\hat{H}_s} = e^{-it\hat{H}_s - i\hat{H}_s^{int}\int_{t_i}^{t} dx\, V(x)} \rho_s\, e^{+it\hat{H}_s + i\hat{H}_s^{int}\int_{t_i}^{t} dx\, V(x)}.$$

To gain a deeper understanding, we first note that since $\hat{H}_s$ and $\hat{H}_s^{int}$ are Hermitian and commute, there exists a mutually orthonormal basis, denoted by $\{|\phi_j\rangle\}_{j=1}^{d_s}$, such that

$$\hat{H}_s = \sum_{j=1}^{d_s} E_j |\phi_j\rangle\langle\phi_j|_s, \tag{S.5}$$

$$\hat{H}_s^{int} = \sum_{j=1}^{d_s} \Omega_j |\phi_j\rangle\langle\phi_j|_s, \tag{S.6}$$

where without loss of generality we can confine $\Omega_j \in [-\pi, \pi)$, for $j = 1, 2, 3, \ldots, d_s$. By writing the evolution of the free Hamiltonian in the $\{|\phi_j\rangle\}_{j=1}^{d_s}$ basis

$$\rho(t) = e^{-it\hat{H}_s} \rho_s \, e^{it\hat{H}_s} = \sum_{m,n=1}^{d_s} \rho_{m,n}(t) |\phi_m\rangle\langle\phi_n|,$$

It can then can be written in the form

$$\rho_s(t) = \sum_{m,n=1}^{d_s} \rho_{m,n}(t) \, e^{-i(\Omega_m - \Omega_n) \int_{t_i}^{t} dt \, V(t)} |\phi_m\rangle\langle\phi_n|.$$

Writing the state ρ_{sc}^{id} at a later time in the $\{|\phi_j\rangle\}_{j=1}^{d_s}$ basis, we find

$$\rho_{sc}^{id}(t) = e^{-it\hat{H}_{total}^{id}} \rho_{sc}^{id} \, e^{it\hat{H}_{total}^{id}} = \sum_{m,n=1}^{d_s} \rho_{m,n}(t) \otimes |\Phi_m(t)\rangle\langle\Phi_n(t)|_c,$$

with

$$|\Phi_n(t)\rangle_c := e^{-it(\hat{p}_c + \Omega_n V(\hat{x}_c))} |\Phi\rangle_c.$$

Exercise 2. ***The limit of p Hamiltonian***

In this exercise, we will look at various limits of the finite clock Hamiltonian

$$H_c = \sum_{n=0}^{d-1} n\omega |n\rangle\langle n|_c.$$

(a) *First show that the recurrence time of the clock is $T = \frac{2\pi}{\omega}$.*

Solution The recurrence time of the clock is the time after which it will to its initial state independent of what it was. For $T = \frac{2\pi}{\omega}$, we have

$$e^{-iH_c \frac{2\pi}{\omega}} = e^{-i \sum_{n=0}^{d-1} 2\pi n |n\rangle\langle n|_c} = \mathbb{1}_c.$$

(b) *Let us first consider $d \to \infty$, with $\omega d \to \infty$. What is the limit of the Hamiltonian in this case? What is the physical meaning of the limit?*

Solution The time intervals between time states $\Delta t = \frac{T_0}{d} = \frac{2\pi}{\omega d}$ are tending to 0, so we arrive to continuous time state. The energy spectrum, however, remains discrete (oscillating energies).

(c) *Now suppose that $\omega d = const$. How does your answer from (b) change?*

Solution Now we have the energy spacing tending to 0, and the time interval between time states is constant. Hence, the energy spectrum is continuous (limit of p Hamiltonian), but the time states are discrete.