**Exercise 1.** ***The quantum pigeonhole principle [1]***

*It is a truth universally acknowledged, that a person in a possession of three pigeons and two holes, will observe one hole occupied by at least two pigeons. However, this is not true for pigeons who are under the command of quantum mechanics. In this exercise, we will consider instances when three quantum particles are put in two boxes, yet no two particles are in the same box, generalize the result, and discuss implications of the result for our understanding of quantum correlations.*

*Consider three particles and two boxes, denoted L (left) and R (right). To start our experiment, we prepare each particle in a superposition of being in the two boxes,*

$$|\Psi\rangle_j = \frac{1}{\sqrt{2}}\left(|L\rangle_j + |R\rangle_j\right),\ j \in \{1,2,3\}.$$

(a) *Write down projectors corresponding to the subspaces of: particles 1 and 2 being in the same box $\Pi_{1,2}^{same}$; and them being in different boxes $\Pi_{1,2}^{diff}$. What are the probabilities of these events given the initial state $|\Psi\rangle = |\Psi\rangle_1|\Psi\rangle_2|\Psi\rangle_3$?*

**Solution** Particles 1 and 2 being in the same box means the state being in the subspace spanned by $|L\rangle_1|L\rangle_2$ and $|R\rangle_1|R\rangle_2$; being in different boxes corresponds to the complementary subspace, spanned by $|L\rangle_1|R\rangle_2$ and $|R\rangle_1|L\rangle_2$. The projectors corresponding to these subspaces are

$$\Pi_{1,2}^{same} = \Pi_{1,2}^{LL} + \Pi_{1,2}^{RR}$$
$$\Pi_{1,2}^{diff} = \Pi_{1,2}^{LR} + \Pi_{1,2}^{RL},$$

where

$$\Pi_{1,2}^{LL} = |L\rangle_1|L\rangle_2\langle L|_1\langle L|_2 \qquad \Pi_{1,2}^{RR} = |R\rangle_1|R\rangle_2\langle R|_1\langle R|_2,$$
$$\Pi_{1,2}^{LR} = |L\rangle_1|R\rangle_2\langle L|_1\langle R|_2 \qquad \Pi_{1,2}^{RL} = |R\rangle_1|L\rangle_2\langle R|_1\langle L|_2.$$

Given just the initial state, the probabilities are

$$P(same) = \langle\Psi|\Pi_{1,2}^{same}|\Psi\rangle = \langle\Psi|\Pi_{1,2}^{LL} + \Pi_{1,2}^{LL}|\Psi\rangle = \frac{1}{2}$$
$$P(diff) = \langle\Psi|\Pi_{1,2}^{diff}|\Psi\rangle = \langle\Psi|\Pi_{1,2}^{LR} + \Pi_{1,2}^{RL}|\Psi\rangle = \frac{1}{2}.$$

*Now, given the same initial state $|\Psi\rangle$, let us measure whether each particle is in the state $|+i\rangle = \frac{1}{\sqrt{2}}(|L\rangle + i|R\rangle)$ or $|-i\rangle = \frac{1}{\sqrt{2}}(|L\rangle - i|R\rangle)$. The cases we are interested in are those in which all particles are found in $|+i\rangle$, i.e. the final state*

$$|\Phi\rangle = |+i\rangle_1|+i\rangle_2|+i\rangle_3.$$

*Neither the initial state nor the finally selected state contain any correlations between the position of the particles. Furthermore, both the preparation and the post-selection are done independently, acting on each particle separately.*

(b) *What are the probabilities of finding the particles 1 and 2 in the same or different boxes at the intermediate time, given the final (post-selected) state $|\Phi\rangle$, and the initial (pre-selected) state $|\Psi\rangle$? Explain why the result can be applied to all pairs of particles.*

**Solution** Suppose that at the intermediate time we find the particles in the same box. The wave function then collapses (up to normalisation) to

$$|\Psi'\rangle = \Pi_{1,2}^{same}|\Psi\rangle = \frac{1}{2}\big(|L\rangle_1|L\rangle_2 + |R\rangle_1|R\rangle_2\big)|+\rangle_3$$

which is orthogonal to the post-selected state, i.e.

$$\langle\Phi|\Pi_{1,2}^{same}|\Psi\rangle = 0.$$

Hence in this case the final measurements cannot find the particles in the state $|\Phi\rangle$. Therefore the only cases in which the final measurement can find the particles in the state $|\Phi\rangle$ are those in which the intermediate measurement found that particles 1 and 2 are in different boxes.

The states we pre- and post-select on are symmetrical under the permutation of particles; hence, the same conclusion can be reached for all pairs of particles, namely, particles 2 and 3, and particles 1 and 3 are also in different boxes.

*To sum up, given the above pre- and post-selection, we have three particles in two boxes, yet no two particles can be found in the same box – our quantum pigeonhole principle.*

(c) *How the principle changes if we post-select on the states $|\Phi\rangle = |-i\rangle_1|-i\rangle_2|-i\rangle_3$ or $|\Phi\rangle = |+i\rangle_1|-i\rangle_2|-i\rangle_3$?*

**Solution** In the case when the final state is $|-i_1|-i_2|-i_3$ the intermediate measurements exhibit once again the quantum pigeon-hole effect, i.e. no two particles can be found in the same box:

$$\langle\Phi|\Pi_{1,2}^{same}|\Psi\rangle = 0.$$

If we assume the final state to be $|+i\rangle_1|-i\rangle_2|-i\rangle_3$, then intermediate measurements find that particle 2 is in the same box as the particle 1, particle 3 is in the same box as 1 but particles 2 and 3 are not in the same box:

$$\langle\Phi|\Pi_{1,2}^{diff}|\Psi\rangle = 0$$
$$\langle\Phi|\Pi_{1,3}^{diff}|\Psi\rangle = 0$$
$$\langle\Phi|\Pi_{2,3}^{same}|\Psi\rangle = 0.$$

(d) *Now consider $N$ particles in $M$ boxes. Show that we can guarantee that no two particles are in the same box when we prepare each particle in the state*

$$|0\rangle_j = \frac{1}{\sqrt{M}}\sum_{k=1}^{M}|k\rangle_j,\ j\in\{1,\dots,N\},$$

*and post-select on each particle being in the state*

$$|\eta\rangle_j = \frac{1}{\sqrt{M}}\sum_{k=1}^{M}e^{\frac{i\pi k}{M}}|k\rangle_j,\ j\in\{1,\dots,N\}.$$

**Solution** The projector corresponding to the particles 1 and 2 are found in the same box is now

$$\Pi_{1,2}^{same} = \sum_{n=1}^{M} |n\rangle_1 |n\rangle_2 \langle n|_1 \langle n|_2.$$

Again, assuming that the state collapses to $\Pi_{1,2}^{same}|\Psi\rangle$, we calculate the overlap with the post-selected state $|\Phi\rangle$:

$$\langle\Phi|\Pi_{1,2}^{same}|\Psi\rangle = \langle\eta|_1\langle\eta|_2 \left(\sum_{n=1}^{M} |n\rangle_1 |n\rangle_2 \langle n|_1 \langle n|_2\right) |0\rangle_1|0\rangle_2 = \frac{1}{M^2}\sum_{n=1}^{M} e^{\frac{-2i\pi n}{M}} = 0.$$

(e) *Coming back to the case of three particles and two boxes, compare the probabilities of finding two particles in the same box for the "pre- selected only" experiment, and for the "pre- and post-selected" one. What does the conclusion tell you about the difference between correlations that can be observed when we measure particles separately and when we measure them jointly?*

**Solution** Neither the pre-selected state nor the post-selected state are correlated (they are both direct products and each particle is prepared and post-selected individually) yet the particles are correlated. If we measure the location of each particle individually, they appear to be completely uncorrelated. Suppose we measure separately the location of particle 1 and 2. There are four possible outcomes of this measurement: $LL$, $LR$, $RL$ $RR$ and, as one can easily show, they all occur with equal probabilities. It is only when we ask *solely about the correlation, and no other information*, (i.e whether the two particles are in the same box or not, without asking in which box they are), that we find them correlated. In some sense, we can see a certain trade-off between the knowledge we can have about the correlation between two systems, and the knowledge we have about their exact – but separate configuration.

### *References*


*[1] Y. Aharonov, F. Colombo, S. Popescu, I. Sabadini, D. C. Struppa, J. Tollaksen, The quantum pigeonhole principle and the nature of quantum correlations, 2014. arXiv:1407.3194.*