Exercise 1. *Pre- and post-selection paradoxes (based on [1])*

Suppose a quantum system is prepared in state $|\psi\rangle$, subjected to an intermediate projective measurement $\mathcal{M} = \{P_j\}$, followed by a final projective measurement that includes the projector onto $|\phi\rangle$ as one of its outcomes. Assuming that no other evolution occurs, the joint probability for obtaining the outcome P_j and passing the post-selection is

$$\mathbb{P}(P_j, \phi|\psi, \mathcal{M}) = |\langle\phi|P_j|\psi\rangle|^2, \tag{1}$$

(a) *Write down the marginal probability for passing the post-selection $\mathbb{P}(\phi|\psi, \mathcal{M})$. Show that the probability for the intermediate measurement conditioned on both the pre- and post-selection can be written as*

$$\mathbb{P}(P_j|\psi, \mathcal{M}, \phi) = \frac{\mathbb{P}(P_j, \phi|\psi, \mathcal{M})}{\mathbb{P}(\phi|\psi, \mathcal{M})} = \frac{|\langle\phi|P_j|\psi\rangle|^2}{\sum_k |\langle\phi|P_k|\psi\rangle|^2}. \tag{2}$$

Solution The marginal probability for passing the post-selection is

$$\mathbb{P}(\phi|\psi, \mathcal{M}) = \sum_j \mathbb{P}(P_j, \phi|\psi, \mathcal{M}) = \sum_j |\langle\phi|P_j|\psi\rangle|^2. \tag{S.1}$$

From this, we can calculate the probabilities for the intermediate measurement conditioned on both the pre- and post-selection as

$$\mathbb{P}(P_j|\psi, \mathcal{M}, \phi) = \frac{\mathbb{P}(P_j, \phi|\psi, \mathcal{M})}{\mathbb{P}(\phi|\psi, \mathcal{M})} = \frac{|\langle\phi|P_j|\psi\rangle|^2}{\sum_k |\langle\phi|P_k|\psi\rangle|^2}.$$

Here we shall only consider cases $\mathbb{P}(P|\psi, \{P, \mathbb{1} - P\}, \phi)$ where the intermediate measurement has two outcomes, and abbreviate $\mathbb{P}(P|\psi, \{P, I - P\}, \phi) = \mathbb{P}(P|\psi, \phi)$ and $\mathbb{P}(\phi|\psi, \{P, I - P\}) = \mathbb{P}(\phi|\psi)$.

Now let use define a logical pre- and post-selection (PPS) paradox. Consider a Hilbert space, a choice of pre-selection $|\psi\rangle$ and post-selection $|\phi\rangle$, and a (finite) set of projectors $\mathcal{P}$ that is closed under complements, i.e. if $P \in \mathcal{P}$ then $\mathbb{1} - P \in \mathcal{P}$. Suppose further that the probabilities $\mathbb{P}(P|\psi, \phi)$ are either 0 or 1 for every $P \in \mathcal{P}$ (which is what leads to the terminology "logical").

Now consider the partial boolean algebra generated by $\mathcal{P}$, i.e. the smallest set of projectors $\mathcal{P}'$ that contains $\mathcal{P}$ and satisfies

- *If $P \in \mathcal{P}'$ then $\mathbb{1} - P \in \mathcal{P}'$.*
- *If $P, Q \in \mathcal{P}'$ and $PQ = QP$ then $PQ \in \mathcal{P}'$.*

If we think of projectors as representing propositions, then these conditions ensure that we can take complements and conjunctions of compatible propositions.

Finally, suppose that we try to extend the probability function $f(P) = \mathbb{P}(P|\psi, \phi)$ from $\mathcal{P}$ to $\mathcal{P}'$ such that the following algebraic conditions *are satisfied*

(i) *For all $P \in \mathcal{P}'$, $0 \le f(P) \le 1$.*
(ii) *$f(I) = 1, f(0) = 0$.*
(iii) *For all $P, Q \in \mathcal{P}'$ such that $PQ = QP$, $f(P + Q - PQ) = f(P) + f(Q) - f(PQ)$.*

If it is not possible to do this then we say that the ABL predictions for $\mathcal{P}$ form a logical PPS paradox.

(b) *Show that the quantum pigeonhole principle is a logical PPS paradox.*

Solution For the analysis of the quantum pigeonhole principle, see Exercise Sheet 11. We consider the projectors

$$P_{1,2}^{same} = P_{1,2}^{LL} + P_{1,2}^{RR}$$
$$P_{1,2}^{diff} = P_{1,2}^{LR} + P_{1,2}^{RL},$$

corresponding to the particles 1 and 2 being in the same or different boxes. The similar projectors can be introduced for particles 2 and 3, and 1 and 3. The finite set of projectors $\mathcal{P} = \{P_{i,j}^{same}, P_{i,j}^{diff}\}$.

The states we pre- and post-select on are

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|L\rangle_1 + |R\rangle_1\right) \otimes \frac{1}{\sqrt{2}}\left(|L\rangle_2 + |R\rangle_2\right) \otimes \frac{1}{\sqrt{2}}\left(|L\rangle_3 + |R\rangle_3\right);$$
$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|L\rangle_1 + i|R\rangle_1\right) \otimes \frac{1}{\sqrt{2}}\left(|L\rangle_2 + i|R\rangle_2\right) \otimes \frac{1}{\sqrt{2}}\left(|L\rangle_3 + i|R\rangle_3\right).$$

Calculating the probabilities, we obtain

$$f(P_{1,2}^{same}) = f(P_{2,3}^{same}) = f(P_{1,3}^{same}) = 0;$$
$$f(P_{1,2}^{diff}) = f(P_{2,3}^{diff}) = f(P_{1,3}^{diff}) = 0.$$

One can also show that $f(P_{1,2}^{diff} P_{2,3}^{diff}) = 0$. Then, according to (iii),

$$f(P_{1,2}^{diff} + P_{2,3}^{diff} - P_{1,2}^{diff} P_{2,3}^{diff}) = f(P_{1,2}^{diff}) + f(P_{2,3}^{diff}) - f(P_{1,2}^{diff} P_{2,3}^{diff}) = 2,$$

in violation with (ii).

(c) *Show that Hardy's paradox is also an example of a logical PPS paradox.*

Solution In Hardy's setting, we pre- and post-select on the following states

$$|\psi\rangle = \frac{1}{\sqrt{3}}|out\rangle_+|in\rangle_- + \frac{1}{\sqrt{3}}|in\rangle_+|out\rangle_- + \frac{1}{\sqrt{3}}|out\rangle_+|out\rangle_-;$$
$$|\phi\rangle = \frac{1}{2}\left(\, |out\rangle_+ - |in\rangle_+\right) \otimes \left(\, |out\rangle_- - |in\rangle_-\right).$$

The set of projectors $\mathcal{P}$ in this case includes the single-particle "occupation" operators $\{|out\rangle\langle out|_+, |in\rangle\langle in|_+, |out\rangle\langle out|_-, |in\rangle\langle in|_-\}$ and their complements, as well as the pair occupation operators. Then we obtain

$$f(|in\rangle\langle in|_+) = 1, \quad f(|in\rangle\langle in|_-) = 1 f(|in\rangle\langle in|_+ |in\rangle\langle in|_-) = 0,$$

which is in contradiction with (ii) again:

$$f(|in\rangle\langle in|_+ + |in\rangle\langle in|_- - |in\rangle\langle in|_+ |in\rangle\langle in|_-) = 1 + 1 - 0 = 2.$$

(d) *Consider another setting: a so-called three-box paradox. It involves a state space spanned by $\{|1\rangle, |2\rangle, |3\rangle\}$ representing a ball in box 1, 2, or 3 respectively. Consider a pre-selection $|\psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$ and a post-selection $|\phi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - |3\rangle)$. Compare the probabilities of finding the ball in the 1st and 2nd boxes. Show that this is also a case of a PPS paradox.*

Solution If we "look in box 1", $\mathcal{M} = \{|1\rangle\langle 1|, |2\rangle\langle 2| + |3\rangle\langle 3|\}$, then whenever the post-selection succeeds we will have found the ball, $\mathbb{P}(|1\rangle\langle 1||\psi, \mathcal{M}, \phi) = 1$. But if instead we "look in box 2", $\mathcal{M}' = \{|1\rangle\langle 1| + |3\rangle\langle 3|, |2\rangle\langle 2|\}$, we also have $\mathbb{P}(|2\rangle\langle 2||\psi, \mathcal{M}', \phi) = 1$. Hence the ball is in both boxes. In terms of algebraic conditions, this means we have

$$f(|1\rangle\langle 1|) = 1,\ f(|2\rangle\langle 2|) = 1.$$

Then according to the condition (iii) it follows $f(|1\rangle\langle 1| + |2\rangle\langle 2|) = f(|1\rangle\langle 1|) + f(|2\rangle\langle 2|) = 2$, in violation with the condition (ii).

Exercise 2. ***Pre- and post-selection paradoxes with weak values***

As we have established previously, an observable A can be measured by coupling the system to a continuous variable pointer system via a Hamiltonian $H = gA \otimes p$, where g is the coupling constant, A is the observable to be measured, and p is the momentum of the pointer. If the parameters are chosen such that $gt \ll \Delta x$, where t is the duration of the measurement interaction and Δx is the initial position uncertainty of the pointer, then this is called a "weak measurement".

If the system is pre- and post-selected, with a weak measurement in the middle, then, to first order in gt, the position distribution of a suitably prepared pointer simply shifts by an amount $gtw(A|\psi, \phi)$, where

$$w(A|\psi, \phi) = \mathrm{Re}\left(\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}\right), \tag{3}$$

and $w(A|\psi, \phi)$ is called the weak value *of A. Weak values can lie outside the eigenvalue range of the operator A, in which case they are called* anomalous *weak values.*

(a) Check that the weak values assigned to a partial boolean algebra of projection operators always satisfy the algebraic conditions (ii) and (iii) in Exercise 1 with $f(P) = w(P|\psi, \phi)$.

Solution Checking the condition (ii), we trivially obtain

$$f(\mathbb{1}) = \mathrm{Re}\left(\frac{\langle\phi||\psi\rangle}{\langle\phi|\psi\rangle}\right) = 1,\ f(0) = \mathrm{Re}\left(\frac{\langle\phi|0|\psi\rangle}{\langle\phi|\psi\rangle}\right) = 0.$$

The condition (iii) also holds:

$$f(P + Q - PQ) = \mathrm{Re}\left(\frac{\langle\phi|(P + Q - PQ)|\psi\rangle}{\langle\phi|\psi\rangle}\right) = f(P) + f(Q) - f(PQ).$$

(b) Verify $\mathbb{P}(P|\psi, \phi)$ is 0 or 1 then $w(P|\psi, \phi) = \mathbb{P}(P|\psi, \phi)$. What does this mean for weak values of weak measurement versions of logical PPS paradoxes?

Solution If $\mathbb{P}(P|\psi, \phi) = 0$, this means that $\langle\phi|P|\psi\rangle = 0$, and $w(P|\psi, \phi) = 0$. If $\mathbb{P}(P|\psi, \phi) = 1$, this means that $|\langle\phi|P|\psi\rangle|^2 = |\langle\phi|P|\psi\rangle|^2 + |\langle\phi|\mathbb{1} - P|\psi\rangle|^2$, and $\langle\phi|\mathbb{1} - P|\psi\rangle = 0$. Hence, $\langle\phi|\psi\rangle = \langle\phi|P|\psi\rangle$, and $w(P|\psi, \phi) = 1$.

Whenever there is a logical PPS paradox for $\mathcal{P}$, there is some projector in the partial boolean algebra $\mathcal{P}'$ that has an anomalous weak value. This is because, by definition, there is no extension of the probabilities to $\mathcal{P}'$ that satisfies all of the algebraic conditions, and condition (i) is the only one that can be violated by weak values. Therefore, logical PPS paradoxes will always show up as anomalous weak values in the weak measurement version of the experiment. For example, in the three-box paradox, we have $w(|1\rangle\langle 1| + |2\rangle\langle 2||\psi, \phi) = 2$ and $w(|3\rangle\langle 3||\psi, \phi) = 1$.

References

[1] Matthew F. Pusey, Matthew S. Leifer. Logical pre- and post-selection paradoxes are proofs of contextuality. EPTCS 195, pp. 295-306 (2015). arXiv:1506.07850