Exercise 1. Practicing Dirac notation: finite Hilbert spaces

Consider a three-dimensional Hilbert space with an orthonormal basis $|1\rangle, |2\rangle, |3\rangle$. Using $a, b, c \in \mathbb{C}$, let us define the states:

$$|\psi\rangle = a|1\rangle - b|2\rangle + c|3\rangle, \qquad |\phi\rangle = b|1\rangle + a|2\rangle.$$

(a) Write down $\langle\phi|$ and $\langle\psi|$. Check that $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$.

(b) What conditions should a, b and c satisfy such that both states are normalized, i.e. $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1$?

(c) Express $|\phi\rangle$ and $|\psi\rangle$ as column vectors. Calculate their inner product.

(d) Let $Q = |\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|$. Is Q hermitian? What are its eigenvalues?

Exercise 2. A particle in a box: continuous Hilbert spaces

Let $\mathcal{H}$ be an infinite dimensional (continuous) Hilbert space, and let $\{|x\rangle\}_{x\in\mathbb{R}}$ and $\{|p\rangle\}_{p\in\mathbb{R}}$ be two orthonormal bases. We can expand any state $|\psi\rangle$ in one of these bases, e.g. $|\psi\rangle = \sum_{x\in\mathbb{R}} \psi(x)|x\rangle$. We call the function $\psi(x)$ the position wave function of the state. Here, we are interested in "conjugate bases", i.e. two bases related by a Fourier transform,

$$|p\rangle = \beta \int_{-\infty}^{\infty} e^{-i\alpha px}|x\rangle dx$$

For reasons that we will see later, we call $\{|x\rangle\}_{x\in\mathbb{R}}$ the position basis, and $\{|p\rangle\}_{p\in\mathbb{R}}$ the momentum basis.

(a) Find the coefficients α, β which normalize the state.

(b) Show that the position and the momentum bases are mutually unbiased, that is,

$$|\langle x|p\rangle|^2 = |\langle x'|p'\rangle|^2 \qquad \forall x, x', p, p' \in \mathbb{Z}.$$

As a more concrete example, let us consider a quantum particle confined in a box of length L with impenetrable walls, the centre of the wall being at the origin of the coordinates. In the ground state, the wave function of the particle is given by [1]:

$$\psi_1(x) = \begin{cases} \gamma\cos(\frac{\pi x}{L}) & \text{if } -\frac{L}{2} < x < \frac{L}{2} \\ 0 & \text{otherwise} \end{cases}$$

(c) Normalize the wave function to determine the coefficient γ.

(d) Write down the wave function in the momentum basis.

[1] the exact derivation will be given later in the course.

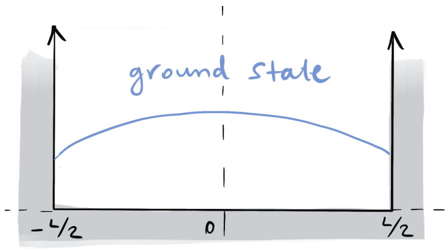

Figure 1: The ground state position wavefunction of a particle in a box.

(e) Suppose that we want to perform a measurement of the position of of the particle. What is the expected value of the position x? What is the expected value of the momentum p if we choose to measure it instead? It is most likely to find the particle in the middle of the box. Analogously, working in the momentum basis we obtain:

$$\langle p\rangle = \int_{-\infty}^{\infty} |\tilde{\psi}_1(p)|^2 p dp = \int_{-\infty}^{\infty} \frac{\pi^2 L}{(\pi^2 - p^2L^2)^2} p \cos^2 \frac{pL}{2} dp = 0.$$

Exercise 3. Bloch sphere and measurements

In this exercise you will be introduced to basic single-qubit gates. The basic gates you will need are (given in matrix form in the computational Z basis):

- Pauli-X, Pauli-Y and Pauli-Z:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{1}$$

- Rotation around the x/y/z - axis by an angle θ:

$$R_x(\theta) = e^{-i\theta X/2} = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}, \quad R_y(\theta) = e^{-i\theta Y/2} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix},$$
$$R_z(\theta) = e^{-i\theta Z/2} = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}, \tag{2}$$

You may already know that any pure state of a qubit can be described and visualised as a point on a 3D ball of radius 1: this is the so-called *Bloch sphere.* In this question we will familiarise ourselves with this representation.

(a) Show that any pure state $|\psi\rangle$ of a single qubit can be written as $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. These are the polar coordinates of state $|\psi\rangle$ on the Bloch sphere.

(b) Write down the the (θ, ϕ) coordinates for each of the following states. Also draw all the state vectors on the Bloch sphere (a state vector is a vector from the centre of the co-ordinate system to the point representing the state).

$$|0\rangle \quad ; \quad |1\rangle \quad ; \quad |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \quad ; \quad |\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle) \ .$$

These states are the eigenstates of Pauli Z, X and Y operators, respectively.

(c) Show that implementing $R_z(\alpha)$ (as given in (2)) corresponds to a rotation by the angle α around the z-axis.

(d) Let us now discuss measurements. Recall that any Hermitian operator M can correspond to a quantum measurement — the measurement basis is the eigenbasis of M. We perform a Z measurement on a state with Bloch coordinates (θ, ϕ). This is called the *computational basis*. What are the possible outcomes, their probabilities and possible post-measurement states? And for an X measurement?

(e) A qubit synthesiser in a quantum factory is supposed to reliably produce states $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. However, an unconfirmed report suggests the machine fails to produce $|+\rangle$. Instead, it randomly produces either $|0\rangle$ (50% of the time) or $|1\rangle$ (also 50% of the time). A quantum mechanic is sent to investigate. He is allowed to perform measurements on the outcoming qubits. How shall the quantum mechanic debug the machine?