Exercise 1. Unitary and Hermitian operators

Let $\mathcal{H}$ be a Hilbert space and $A, B \in End(\mathcal{H}, \mathcal{H})$ operators in that Hilbert space. Here you have to prove some of their properties.
Note: the operator exponential is given by the power series:

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

(a) Show that $(e^A)^\dagger = e^{A^\dagger}$.

(b) Suppose that $[A, B] = AB - BA = 0$, that is, the operators A and B commute. Prove that $e^{A+B} = e^A e^B$.

(c) Show that if the operator A is Hermitian ($A = A^\dagger$) , then $U = e^{iA}$ is unitary ($UU^\dagger = U^\dagger U = \mathbb{I}$). Show also that for a collection $\{A_j\}_j$ of Hermitian operators , $U = \bigotimes_j e^{iA_j}$ is unitary.

Hint: Make use of the results in (a) and (b).

(d) Show that if U is a unitary, then there exists a Hermitian operator A such that $U = e^{iA}$.

(e) Suppose that V is both unitary and Hermitian. Show that the only possible eigenvalues fro V are ± 1 and that $V^2 = \mathbb{I}$.

(f) Suppose that an observable $A = A(t)$ is time-dependent, and H is the Hamiltonian of the system. Show that

$$\frac{d}{dt}\langle A \rangle = \frac{1}{i\hbar}\langle \psi | [A, H] | \psi \rangle + \langle \psi | \frac{dA}{dt} | \psi \rangle$$

Exercise 2. Elitzur-Vaidman bomb test

Suppose that you have a box that is either empty or contains a very sensitive bomb that would explode if hit by only a single photon of light. You have to determine whether the box is empty or not (while staying alive). This can be done with high probability using a trick known as interaction-free measurement.

Let us model a photon as a two-level quantum system, e.g. a qubit. To carry out the test, we need a Mach-Zehnder interferometer like the one we saw last week. The difference is that now we use beam splitters that modify the state of the photon as given by the unitary $U(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ in the computational basis $\{|0\rangle, |1\rangle\}$ ($|0\rangle$ and $|1\rangle$ can be understood as corresponding to transmitted and reflected beams). To test whether the box contains a bomb, you can make two tiny holes at its opposite sides such that only a single photon could pass through, and place it on one of the two paths of the interferometer, say the bottom one.
The photon starts out in the state $|0\rangle$. It then passes through a $\phi = \pi/4$ beam splitter which

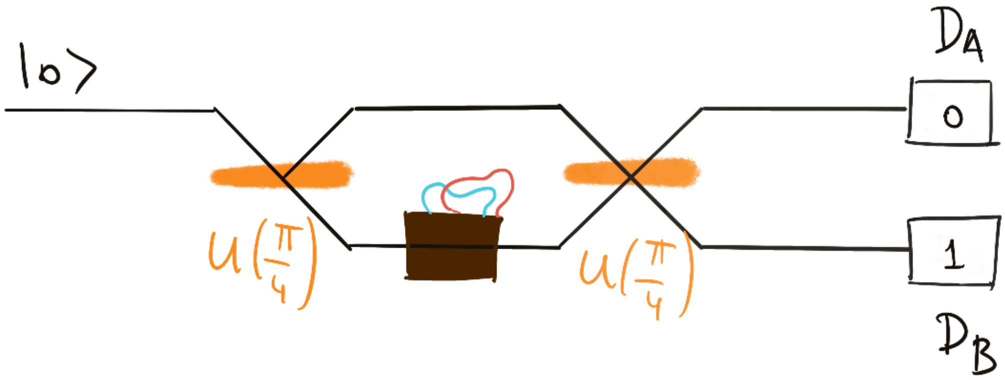

Figure 1: The experimental setup of Elitzur-Vaidman bomb tester.

sends it onto two possible paths in superposition. The bottom path goes through the box, potentially resulting in a measurement due to the bomb. The two paths are then recombined by another $\phi = \pi/4$ beam splitter, and the photon is measured at the end by the detectors D_A, D_B in the $\{|0\rangle, |1\rangle\}$ basis by a projective measurement.

(a) Assuming there was no bomb inside the box, derive the state before the final measurement.

(b) Assuming there was a bomb inside the box, what is the probability of triggering the bomb?

(c) Assuming there was a bomb and it did not explode, derive the state before the final measurement.

(d) Assume you run the experiment without knowing if the bomb is present. If no explosion occurred, what can you conclude from the outcome of the final measurement? What do you learn if the outcome was 1 and what do you learn if the outcome was 0?

Now let the initial state be $|0\rangle$ as before. For some integer $n \geq 1$ repeat the following two steps n times: apply the beam splitter $U(\pi/(2n))$ and let the the bottom path go through the box.

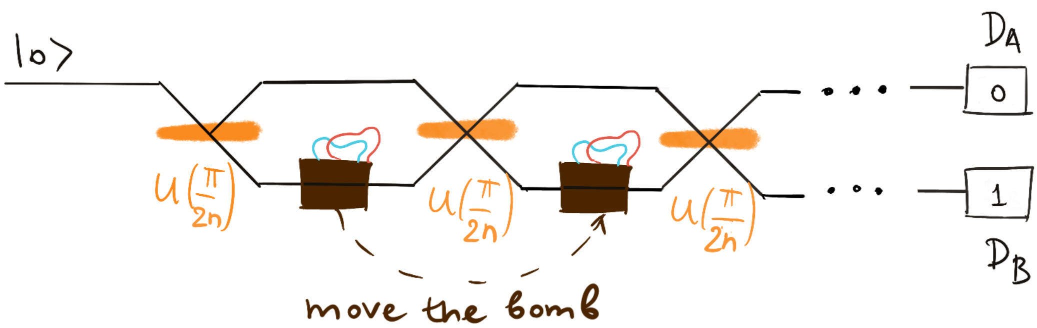

Figure 2: The experimental setup of Elitzur-Vaidman bomb tester with n copies of beam splitter (after each step we move the box further along the interferometer).

(e) What is the final state if there was no bomb?

(f) What is the final state if there was a bomb and it did not explode in any of the n trials?

(g) Assuming the bomb is present, compute the probability of you still being alive at the end of the full test as a function of n. Additionally, compute the probability of guessing correctly (conditioned on you being alive) after the test. What value of n should you choose so that you are still alive with probability 99.9%? And to guess correctly with probability 95%?