Exercise 1. Wave function in 3D

In the lecture, we have seen how a wave function can be introduced in the three-dimensional space $\psi(\vec{r})$.

(a) Show that the momentum wave function in 3D can be written as

$$\bar{\psi}(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint d\vec{r} e^{-i\vec{p}\cdot\vec{r}/\hbar}\psi(\vec{r})$$

(b) We can define operators corresponding to P_X, P_Y, P_Z in 3D as tensor products including 1D momentum operators: for example, $P_X := P_X \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z$. Show that they act on the wave position function in a following way:

$$P_X \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z : \psi(\vec{r}) \to -i\hbar\frac{\partial}{\partial x}\psi(\vec{r})$$
$$\mathbb{I}_X \otimes P_Y \otimes \mathbb{I}_Z : \psi(\vec{r}) \to -i\hbar\frac{\partial}{\partial y}\psi(\vec{r})$$
$$\mathbb{I}_X \otimes \mathbb{I}_Y \otimes P_Z : \psi(\vec{r}) \to -i\hbar\frac{\partial}{\partial z}\psi(\vec{r})$$

(c) Consider commutation relations among X, Y, Z, P_X, P_Y, P_Z. Show that all of the commutators are zero except for:

$$[X, P_X] = [Y, P_Y] = [Z, P_Z] = i\hbar\mathbb{I}$$

(d) Let us define $P^2 = P_X^2 + P_Y^2 + P_Z^2 := P_X^2 \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z + \mathbb{I}_X \otimes P_Y^2 \otimes \mathbb{I}_Z + \mathbb{I}_X \otimes \mathbb{I}_Y \otimes P_Z^2$. Show that P^2 acts on the wave position and momentum functions in following ways:

$$P^2 : \psi(\vec{r}) \to -\hbar^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\psi(\vec{r})$$
$$P^2 : \bar{\psi}(\vec{p}) \to (P_X^2 + P_Y^2 + P_Z^2)\bar{\psi}(\vec{p})$$

Exercise 2. Parity operator

Consider an operator $\mathcal{P}$ in 1D which represents the reflection about the point $x = 0$ (it is called the *parity* operator):

$$\mathcal{P} : \psi(x) \to \psi(-x), \quad \mathcal{P}|x\rangle = |-x\rangle$$

(a) Find the continuous matrix representation $\mathcal{P}(x, x')$ for the parity operator.

(b) Show that the only possible eigenvalues of $\mathcal{P}$ are ± 1. What would it mean to measure $\mathcal{P}$ in the lab?

Exercise 3. Gaussian wave packets

Wave packets with various specific envelope functions $\phi(x)$ can be constructed. A common choice is the Gaussian wave packet, in which the envelope function has a Gaussian form:

$$\psi(x) = \phi(x)e^{ikx} = \frac{1}{\sqrt[4]{2\pi\sigma^2}} \exp(-\frac{(x-x_0)^2}{4\sigma^2}) \cdot e^{ikx}$$

(a) Determine the momentum wave function.

(b) What are expected values of x and p?

(c) Calculate variances $\langle \Delta x^2 \rangle$ and $\langle \Delta p^2 \rangle$.

(d) For which values of x_0 and σ is the Gaussian wave packet an eigenstate of the parity operator $\mathcal{P}$? What is the associated eigenvalue?

Exercise 4. Entanglement and teleportation

Imagine that Alice (A) has a pure state $|\phi\rangle_S$ of a system S in her lab. She wants to send that state to Bob, who lives, of course, on the Moon, but she does not trust the postwoman Eve to carry it there personally. Here, we will see that if Alice and Bob share an entangled state Alice can "teleport" the state $|\phi\rangle$ to the system B that Bob controls.

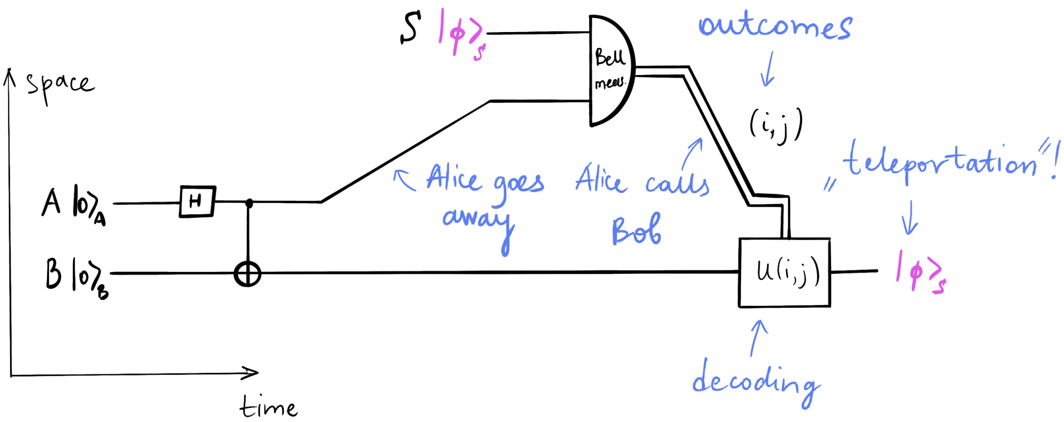

Figure 1: The quantum circuit for the quantum state teleportation.

Formally, we have three systems $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. In this exercise we will assume all three are qubits. The initial state is

$$|\phi\rangle_S \otimes \frac{1}{\sqrt{2}} (|00\rangle_{AB} + |11\rangle_{AB}) \,, \tag{1}$$

i.e. S is decoupled from A and B and these two are fully entangled in a Bell state (they can create this state by starting in states $|0\rangle_A$ and $|0\rangle_B$ and applying Hadamard and CNOT gates in sequence, as shown on the Figure 1). We may write $|\phi\rangle_S = \alpha|0\rangle_S + \beta|1\rangle_S$.

(a) In a first step, Alice will measure systems S and A jointly in the Bell basis, that is, the projectors that represent different outcomes are $\{|\psi_i\rangle\langle\psi_i|_{SA}\}_{i=1,..,4}$, where

$$\tfrac{1}{\sqrt{2}} (|00\rangle_{SA} + |11\rangle_{SA}) \,, \tfrac{1}{\sqrt{2}} (|00\rangle_{SA} - |11\rangle_{SA}) \,, \tfrac{1}{\sqrt{2}} (|01\rangle_{SA} + |10\rangle_{SA}) \,, \tfrac{1}{\sqrt{2}} (|01\rangle_{SA} - |10\rangle_{SA}) \,.$$

Then Alice communicates (classicaly) the result of her measurement to Bob. What is the reduced state of Bob's system (B) for each of the possible outcomes?

(b) Depending on the classical outcomes of the measurement by Alice, Bob needs to apply a combination of two operators from the set $\{X, Z, \mathbb{I}\}$ (where X, Z are Pauli operators). What is this combination?