Exercise 1. Interaction Hamiltonian: coupling to momentum

Consider a system $\mathcal{H}_A \otimes \mathcal{H}_X$, where $\mathcal{H}_A$ represents a discrete subsystem of interest, and $\mathcal{H}_X = \text{span}\{|x\rangle\}_{x\in\mathbb{R}}$ represents the x coordinate of a particle.

Let $M_A = \sum_i a_i |i\rangle\langle i|_A$ be an observable on system A, which we cannot observe directly. We saw in the lecture how to couple this observable to the position of the particle. That is, for an initial state

$$|\tilde{\psi_0}\rangle_{AX} = \left(\sum_i c_i \ |i\rangle_A\right) \otimes |\psi_0\rangle_X = \left(\sum_i c_i \ |i\rangle_A\right) \otimes \int dx \ \psi_0(x) \ |x\rangle_X$$

evolving under the Hamiltonian $H = M_A \otimes P_X$ for a period of time t resulted in the state

$$U_{AX}(t) \ |\tilde{\psi_0}\rangle_{AX} = e^{-\frac{it}{\hbar} M_A \otimes P_X} \ |\tilde{\psi_0}\rangle_{AX} = \sum_i c_i \ |i\rangle_A \otimes \int dx \ \psi(x + t a_i) \ |x\rangle_X,$$

that is, the position wave function shifts by a distance $\Delta x_i = t a_i$ for each term of the superposition. In this exercise, we will couple M_A to a shift in the momentum wave function instead.

(a) Consider the Hamiltonian $H = M_A \otimes \hat{X}_X$. Show that

$$U(t) \ |\tilde{\psi_0}\rangle_{AX} = \sum_i c_i \ |i\rangle_A \otimes e^{\frac{it}{\hbar} a_i \hat{X}_X} \ |\psi_0\rangle_X.$$

(b) Show that the "modular position" operator $e^{-iL\hat{X}/\hbar}$ acts on the position wave function as

$$e^{-iL\hat{X}/\hbar} : \psi(x) \to \psi(x) \ e^{-iLx/\hbar}$$

(c) Use the Fourier transform to show that the modular position acts on the momentum wave function as:

$$e^{-iL\hat{X}/\hbar} : \bar{\psi}(p) \to \bar{\psi}(p + L)$$

(d) For a concrete example, consider the initial state

$$|\psi\rangle_{AX} = (\alpha|0\rangle_A + \beta|1\rangle_A) \otimes |\psi_0\rangle_X,$$

where $|\psi_0\rangle_X$ is a Gaussian state satisfying $\langle x\rangle_0 = 0$, $\Delta x^2 = \sigma^2$, $\langle p\rangle_0 = \hbar k_0$, and the observable $M_A = \gamma Z_A = \gamma|0\rangle\langle 0|_A - \gamma|1\rangle\langle 1|_A$, where γ is a constant that can be adjusted. What is the final state after the evolution under $H = M_A \otimes X_X$ for a time t? What is the sign of γ if we want $|0\rangle$ to be correlated with a particle moving to the left, and $|1\rangle$ with moving to the right?

(e) Repeat the previous calculation for a 2D Gaussian $|\psi_0\rangle_{XY}$ with $\langle x\rangle_0 = \langle y\rangle_0 = 0$, $\Delta x^2 = \Delta y^2 = \sigma^2$, $\langle P_X\rangle_0 = \langle P_Y\rangle_0 = \hbar k_0$ for the evolution with the Hamiltonian $H = \gamma Z_A \otimes X_X \otimes \mathbb{I}_Y$.

Exercise 2. A non-local game: the GHZ setup

In the previous exercise sheet, we have seen that entanglement is a useful resource for a task such as the state teleportation. Here we will look at yet another aspect of entanglement, namely, at how parties (Alice, Bob and Charlie) sharing an entangled state can use correlations between their measurement outcomes. These correlations can be studied by means of non-local games, which obey a following scheme: the referee chooses a triple of questions (r, s, t) (according to some prespecified distribution), sends r to Alice, s to Bob, and t to Charlie, and they answer with a, b and c respectively. The referee evaluates some predicate on (r, s, t, a, b, c) to determine if they win or lose.

Let us look at an example of such a game (called the GHZ game). The referee chooses a three bit string rst uniformly from the set $\{000, 011, 101, 110\}$. Alice, Bob and Charlie win if $a \oplus b \oplus c = r \vee s \vee t$, and lose otherwise. In other words, they win if $a \oplus b \oplus c = 0$ for the set of questions 000, and if $a \oplus b \oplus c = 1$ otherwise. Table 1 lists the winning condition for each possible set of questions.

rst	$a \oplus b \oplus c$
000	0
011	1
101	1
110	1

Table 1: The winning conditions for GHZ game.

(a) Let us first consider a deterministic strategy, where each answer is a function of the question received (and Alice, Bob and Charlie don't share an entangled state). Denote a_r, b_s and c_t the answers that would be given for each choice of r, s, and t. For example, if $a_0 = 1$ and $a_1 = 0$, then Alice always answers the question 0 with 1 and the question 1 with 0. What are the winning conditions? What is the probability of winning?

(b) We will not prove it here, but indeed no classical strategy, where the players can discuss what to do before hand, can do better than the deterministic strategies discussed above. Now let us turn to quantum strategies. Suppose that Alice, Bob and Charlie share an entangled state:

$$|\psi\rangle_{ABC} = \frac{1}{2}|0\rangle_A|0\rangle_B|0\rangle_C - \frac{1}{2}|0\rangle_A|1\rangle_B|1\rangle_C - \frac{1}{2}|1\rangle_A|0\rangle_B|1\rangle_C - \frac{1}{2}|1\rangle_A|1\rangle_B|0\rangle_C$$

Each player will use the same strategy:

1. if the question is $q = 1$, then the player performs a Hadamard transform on their qubit of the above state (if q = 0, the player does not perform a Hadamard transform).
2. the player measures their qubit in the computational basis $\{|0\rangle, |1\rangle\}$ and returns 0 or 1 to the referee.

Show how this strategy works out. What is the probability of winning this time?