Exercise 1. Evolution of Gaussian wave packet

In this exercise, we will look at the evolution of a Gaussian packet $\psi(x,t)$. Suppose that at $t = 0$ the wave function is characterized as follows:

$$\psi(x,0) = \phi(x)e^{ikx} = \frac{1}{\sqrt[4]{2\pi\sigma^2}} \exp(-\frac{(x-x_0)^2}{4\sigma^2}) \cdot e^{ikx}$$

(a) Show that the initial momentum wave function is given by

$$\bar{\psi}(p,0) = \bar{\phi}(p - \hbar k),$$

where $\bar{\phi}(p)$ is the Fourier transform of $\phi(x)$ (Exercise Sheet 4, Exercise 3).

(b) Show that for a free particle Hamiltonian $H = \frac{P^2}{2\mu}$, the momentum wave function evolves as

$$\bar{\psi}(p,t) = \bar{\psi}(p,0)e^{-\frac{ip^2}{2\mu\hbar}t}$$

(c) Show that

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \bar{\psi}(p,t)\ e^{\frac{i}{\hbar}px}\ dp$$

is a solution of the free particle Schrödinger equation, and for the given case of the Gaussian wave packet verify that

$$\psi(x,t) = \frac{(4\pi\sigma^2)^{1/4}}{2\pi\hbar} \sqrt{\frac{\pi}{\frac{\sigma^2}{\hbar^2} + \frac{it}{2\mu\hbar}}}\ \exp\left(-\frac{(x - x_0 - \frac{\hbar k t}{\mu})^2}{4\sigma^2 + \frac{2it\hbar}{\mu}}\right)\ e^{ikx - \frac{it}{\hbar}\frac{\hbar^2 k^2}{2\mu}}$$

(d) Show that a Gaussian wave packet stays a Gaussian wave packet: show that

$$|\psi(x,t)|^2 = \frac{(4\pi\sigma^2)^{1/2}}{4\pi\hbar^2}\ \frac{1}{\frac{\sigma^4}{\hbar^4} + \frac{t^2}{4\mu^2\hbar^2}}\ \exp\left(-\frac{\sigma^2(x - x_0 - \frac{\hbar k t}{\mu})^2}{2\sigma^4 + \frac{t^2\hbar^2}{2\mu^2}}\right)$$

and plot it for an arbitrary time t.

(e) Determine the expectation values of the position and momentum operators for an arbitrary time t from $\psi(x,t)$, $\langle x\rangle_t$ and $\langle p\rangle_t$, and show that they match the values computed in the lecture through the Heisenberg equation of motion, that is, $\langle x\rangle_t = x_0 + \hbar k t$ and $\langle p\rangle_t = \hbar k$.

Exercise 2. Group velocity and phase velocity

For a general wave packet of the form

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} f(k)\ e^{ikx - i\omega(k)t}\ dk$$

one can distinguish two different notions of velocity. First, the speed at which a point of given phase on the wave propagates through space, which referred to as *phase velocity.* Additionally, one can look at the *group velocity* – the speed at which the envelope function travels through space. In other words, the packets of waves move collectively with the group velocity v_{group}; the crest of a wave within each packet travels at the phase velocity v_{phase}.

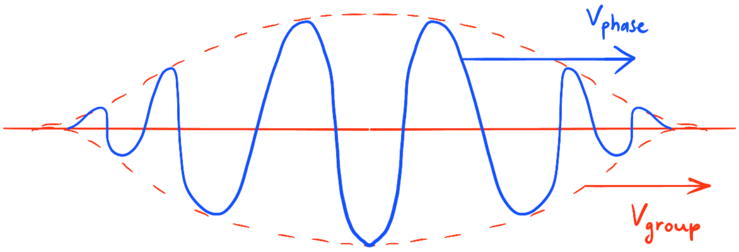

Figure 1: The phase and group velocities of a wave packet.

(a) Suppose, as in the case of the Gaussian wave packet, that $f(k)$ is peaked around a particular value k_0. In that case, we can make a Taylor expansion of $\omega(k)$ around $k = k_0$ (with $\omega_0 = \omega(k_0)$):

$$\omega(k) = \omega_0 + \left(\frac{d\omega}{dk}\right)_{k_0} (k - k_0) + O[(k - k_0)^2]$$

Insert this expansion into $\psi(x, t)$.

(b) Rewrite $\psi(x, t)$ as a product of two waveforms: a plane wave moving with velocity $v_{phase} = \frac{\omega_0}{k_0}$ and a wavepulse moving with velocity $v_{group} = \left(\frac{d\omega}{dk}\right)_{k_0}$. Calculate v_{group} for the case of the free particle ($\omega(k) = \omega(\frac{p}{\hbar}) = \frac{p^2}{2\mu\hbar}$ and $p = \hbar k$).

Hint: *Verify* $\psi(x, t) \approx e^{i(k_0 x - \omega_0 t)} \int f(k)\ \exp[i(k - k_0)(x - v_{group})]\ dk$.

Exercise 3. Measuring the slit

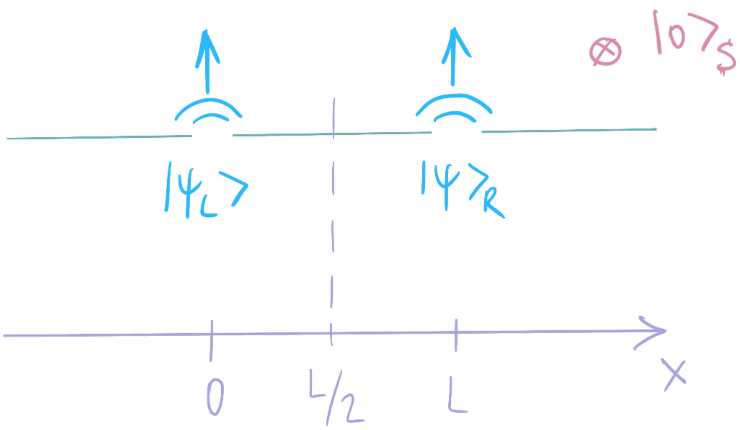

Figure 2: The configuration of the particle immediately after passing the slit, and a pointer particle S which will act as a measurement device, and in this case is a qubit.

We will simulate a measurement of the slit through which an e^- passes in a double-slit experiment. The initial state immediately after passing the slit is:

$$|\psi_\alpha\rangle = \frac{1}{\sqrt{2}}|\psi_L\rangle + \frac{1}{\sqrt{2}}e^{i\alpha}|\psi_R\rangle$$

with $\psi_L(x-L) = \psi_R(x)$. Suppose that each term only has support in a small region which are non-overlapping $\langle\psi_L|\psi_R\rangle = 0$, in particular, $\psi_L^*(x)\psi_R(x) = 0 \ \forall x$. Let us denote

$$\Pi_L = \int_{-\infty}^{L/2} dx\ |x\rangle\langle x|, \quad \Pi_R = \mathbb{I} - \Pi_L = \int_{L/2}^{+\infty} dx\ |x\rangle\langle x|$$

For Π_L and Π_R we have:

$$\begin{aligned} \Pi_L|\psi_L\rangle &= |\psi_L\rangle \qquad \Pi_R|\psi_L\rangle = 0 \\ \Pi_R|\psi_R\rangle &= |\psi_R\rangle \qquad \Pi_L|\psi_R\rangle = 0 \end{aligned}$$

We measure the path by placing a particle next to the slit on the right. This particle has an internal degree of freedom of interest, for example the spin, represented by a qubit Hilbert space $\mathcal{H}_S$. This degree of freedom is sensitive to the presence of the electron: if the particle's spin flips, we know that it detected the electron going through the right slit.

(a) We take the unitary evolution

$$U_{XS} = \Pi_L \otimes \mathbb{I}_S + \Pi_R \otimes \hat{X}_S,$$

where $\hat{X}_S$ is the Pauli X operator. Show that under this evolution the initial joint state of the electron's x component and the particle's spin is mapped as:

$$|\psi_\alpha\rangle_S \otimes |0\rangle_S \xrightarrow{U_{XS}} \frac{1}{\sqrt{2}}|\psi_L\rangle \otimes |0\rangle_S + \frac{1}{\sqrt{2}}e^{i\alpha}|\psi_R\rangle \otimes |1\rangle_S$$

(b) Design an interaction Hamiltonian H_{XS} and corresponding interaction time τ that implement that measurement, i.e. such that $U(\tau) = e^{-iH_{XS}\tau/\hbar} = U_{XS}$.

(c) Show that if we now measure $\mathcal{H}_S$ in the Z basis, the interference pattern disappears, and the electron's observed behaviour is like "a particle".

Hint: Compute $|\psi(x,t)|^2$ for the post measurement state.

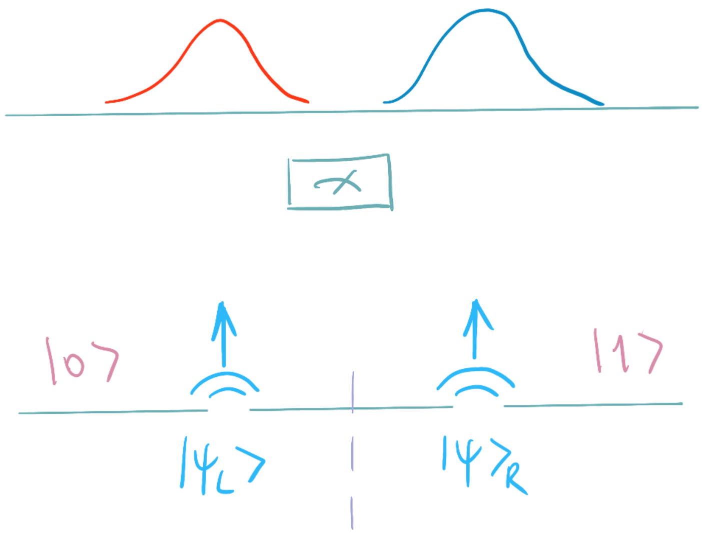

Figure 3: Two distinct measurement results of the internal degree of freedom of the particle after passing the slit.