Exercise 1. A particle in a finite well

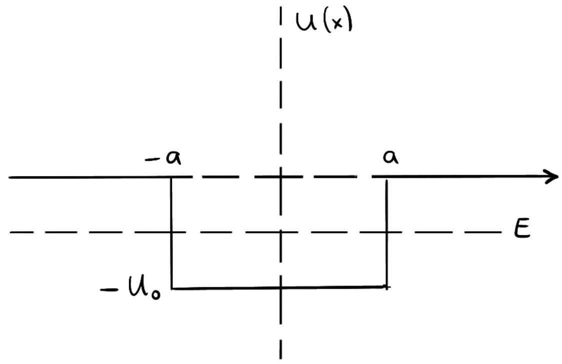

Figure 1: The finite well potential $U(x)$. The energy E is shown as a dashed line in the figure.

In the lecture, we have investigated the behavior of the wave function of the particle confined in a box, which was represented as an infinite well potential. In this exercise we will look at the case of the particle in a *finite* well potential:

$$U(x) = \begin{cases} -U_0 & |x| \leq a, U_0 > 0 \\ 0 & |x| > a \end{cases}$$

We will be interested in bound states namely, energy eigenstates that are normalizable. For this the energy E of the states must be negative. This is readily understood. If $E > 0$, any solutions in the region $|x| > a$ where the potential vanishes would be a plane wave, extending all the way to infinity. Such a solution would not be normalizable. Hence, we have $-U_0 < E < 0$.

Moreover, we can note that the potential $U(x)$ for the finite square well is an even function of x: $U(-x) = U(x)$. We can therefore use that for an even potential the bound states are either symmetric (even solutions) or antisymmetric (odd solutions).

(a) Write down the Schrödinger's equation for the wavefunction of the particle in the potential $U(x)$.

(b) We can find the solution to the resulting differential equation by solving it separately for two regions:

$$\psi(x) = \begin{cases} \psi_1(x) & x < |a| \\ \psi_2(x) & |x| > a \end{cases}$$

Write down the differential equations for ψ_1 and ψ_2 and solve them up to the constants (which will be determined by the boundary conditions), writing out even and odd solutions separately.

(c) The wave function $\psi(x)$ has to be continuously differentiable. What conditions does that impose on constants in $\psi_1(x)$ and $\psi_2(x)$?

(d) Let us introduce unit free constants η, ζ and z_0 as follows:

$$\eta = a\sqrt{\frac{2m}{\hbar^2}(U_0 - |E|)}$$
$$\zeta = a\sqrt{\frac{2m|E|}{\hbar^2}}$$
$$z_0^2 = \frac{2mU_0a^2}{\hbar^2}$$

Rewrite the boundary criteria in terms of these variables for odd and even solutions.

(e) These equations can be solved numerically to find all solutions that exist for a given fixed value of z_0. Each solution represents one bound state. We can understand the solution space by plotting these two equations in the first quadrant of an (η, ζ) plane: do a sketch for the odd and even cases.

Exercise 2. Stationary phase approximation* (extra)

In this exercise, we will justify the stationary phase approximation used in the lecture. It is a mathematical tool used to approximate integrals of the general form [1]:

$$I(t) = \int_{-\infty}^{+\infty} e^{itf(k)} g(k) dk \quad (t \to \infty)$$

To perform the approximation, one needs to find non-degenerate critical points of $f(k)$: all k such that $f'(k) = 0$, but $f''(k) \neq 0$. The idea is then to argue that the most significant contributions to the integral are made in the neighbourhood of such points k. Let us demonstrate this by performing the following calculation.

(a) Consider the integrand in the δ-neighbourhood of an arbitrary point k_0, and assume that we can neglect the change in the $g(k)$ for this interval: $g(k) \approx g(k_0)$. Verify that

$$\int_{k_0-\delta}^{k_0+\delta} e^{itf(k)} g(k)\ dk \ \propto\ \frac{O(1)}{f'(k_0)}$$

Hint: Expand $f(k)$ into its Taylor series in the neighbourhood of k_0: $f(k) \approx f(k_0) + f'(k_0)(k - k_0)$.

This result means that the contribution of the neighbourhood of k_0 is the biggest if $f'(k_0) = 0$.

(b) Now let us consider a function $f(k)$ which has only one non-degenerate critical point k_0. Then it is only the neighbourhood of k_0 that contributes to the integral; expand $f(k)$ up to the second order in the neighbourhood of k_0, and show that

$$I(t) \approx g(k_0)\ e^{itf(k_0) - i\frac{\pi}{4}} \sqrt{\frac{2\pi}{tf''(k_0)}}$$

[1] to be precise, this method works only for "fast enough" functions f, but we will not elaborate on this here.

(c) Derive the result from the lecture: consider the integral describing the wave function of a wave packet which is evolving under the free particle Hamiltonian for a time t

$$\psi(vt, x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{\frac{it}{\hbar}(vp - \frac{p^2}{2\mu})} \; \bar{\phi}(p - p_0)$$

Find the critical point of the function in the exponential. Verify that for $t \to \infty$ it can be approximated to

$$\psi(vt, x) \approx \sqrt{\frac{\mu}{t}} \; \exp\left(it\frac{v^2\mu}{2\hbar} + i\frac{\pi}{4}\right) \; \bar{\phi}(v\mu - p_0)$$