Exercise 1. Stationary states and evolution of general states

(a) Let H be a Hermitian operator on a Hilbert space $\mathcal{H}$. Show that the eigenvectors of H form an orthonormal basis of the Hilbert space.

(b) Express the time evolution of an arbitrary $|\psi\rangle \in \mathcal{H}$ in terms of the evolution of stationary states of the Hamiltonian H.

Hint: Decompose $|\psi\rangle$ as a linear superposition of stationary states which form an orthonormal basis in $\mathcal{H}$.

(c) Now express the expectation value of energy of an arbitrary state $|\psi\rangle$ at time t in terms of eigenvalues of the Hamiltonian.

Exercise 2. Properties of trace

The trace of an operator $A : \mathcal{H} \to \mathcal{H}$ is defined as $\operatorname{tr}(A) = \sum_j \langle j|A|j\rangle$, where $\{|j\rangle\}_j$ is an orthonormal basis in $\mathcal{H}$. Show that the trace operation is:

(a) Linear: $\operatorname{tr}(\alpha A + \beta B) = \alpha \operatorname{tr}(A) + \beta \operatorname{tr}(B)$ for all operators A, B and coefficients $\alpha, \beta \in \mathbb{C}$;

(b) Cyclic: $\operatorname{tr}(ABC) = \operatorname{tr}(BCA)$ for all operators A, B, C;

(c) Basis-independent: $\operatorname{tr}(UAU^\dagger) = \operatorname{tr}(A)$ for all operators A and arbitrary unitaries U.

Exercise 3. Partial trace (in the double slit experiment)

The partial trace is an important concept in the quantum mechanical treatment of multi-partite systems, and it is the natural generalisation of the concept of marginal distributions in classical probability theory. Let ρ_{XS} be a density matrix on the bipartite Hilbert space $\mathcal{H}_X \otimes \mathcal{H}_S$ and $\rho_X = \operatorname{tr}_S(\rho_{XS}) = \sum_j (\mathbb{I}_X \otimes \langle j|_S)\, \rho_{XS}\, (\mathbb{I}_X \otimes |j\rangle_S)$ the marginal on $\mathcal{H}_X$.

(a) Show that ρ_X is a valid density operator by proving it is:

(i) Hermitian: $\rho_X = \rho_X^\dagger$.

(ii) Positive: $\rho_X \geq 0$.

(iii) Normalised: $\operatorname{tr}(\rho_X) = 1$.

Let us return to the double slit experiment. In Exercise Sheet 6, we looked at the interaction of of a particle going through a double slit and a measurement device (modelled as a qubit):

$$\underbrace{\left(\frac{1}{\sqrt{2}}|\psi_L\rangle_X + \frac{e^{i\alpha}}{\sqrt{2}}|\psi_L\rangle_X\right) \otimes |0\rangle_S}_{|\psi'\rangle_{XS}} \xrightarrow{U_{XS}} \underbrace{\frac{1}{\sqrt{2}}|\psi_L\rangle_X \otimes |0\rangle_S + \frac{e^{i\alpha}}{\sqrt{2}}|\psi_R\rangle_X \otimes |1\rangle_S}_{|\psi_0\rangle_{XS}}$$

Suppose that the particle is evolving for some time t after the interaction. The evolution is described by a unitary $U_X(t)$:

$$\underbrace{\frac{1}{\sqrt{2}}|\psi_L\rangle_X \otimes |0\rangle_S + \frac{e^{i\alpha}}{\sqrt{2}}|\psi_R\rangle_X \otimes |1\rangle_S}_{|\psi_0\rangle_{XS}} \xrightarrow{U_X(t)\,\otimes\,\mathbb{I}} \underbrace{\frac{1}{\sqrt{2}}U_X(t)|\psi_L\rangle_X \otimes |0\rangle_S + \frac{e^{i\alpha}}{\sqrt{2}}U_X(t)|\psi_R\rangle_X \otimes |1\rangle_S}_{|\psi(t)\rangle_{XS}}$$

(b) Compute the reduced states after the interaction:

$$\rho_X = \mathrm{tr}_S\ (|\psi_0\rangle\langle\psi_0|_{XS})$$
$$\rho_S = \mathrm{tr}_X\ (|\psi_0\rangle\langle\psi_0|_{XS})$$

(c) Show that the evolution of the reduced state of the particle is given by:

$$\rho_X(t) = U_X(t)\rho_X U_X^\dagger(t) = \mathrm{tr}_S\ (|\psi(t)\rangle\langle\psi(t)|_{XS})$$

Express the evolution in terms of the left and right wavepacket functions, namely, $\rho_X^L = |\psi_L\rangle\langle\psi_L|_X$ and $\rho_X^R = |\psi_R\rangle\langle\psi_R|_X$.

(d) Now suppose that the measurement interaction does not happen. Compute the reduced state of the particle in the beginning $\rho'_X = \mathrm{tr}_S\ (|\psi'\rangle\langle\psi'|_{XS})$ and evolve it as in (b) using $U_X(t)$. What is the difference between $\rho'_X(t)$ and $\rho_X(t)$? What can you conclude about the (lack of) interference fringes?