Exercise 1. Density matrices in the Bloch sphere

We already got an introduction to the Bloch sphere. However, so far, we were only looking at pure states. Now we will also consider mixed states.
In this exercise we will see that any density operator of a qubit can be written as

$$\rho = \frac{1}{2}(\mathbb{I} + \vec{r} \cdot \vec{\sigma}), \tag{1}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices and $\vec{r} = (r_x, r_y, r_z) \in \mathbb{R}^3, |\vec{r}| \leq 1$ is the so-called Bloch vector that gives us the position of a point in a unit ball. The surface of that ball is usually known as the Bloch sphere. Thereby we go through a few properties of density matrices describing states of a qubit using the Bloch representation. This way of expressing qubit states is very convenient and frequently used in quantum information theory.

Hint: The (anti-)commutation relations for Pauli matrices will be helpful.

(a) Using Eq. (1):

  (i) Find and draw in the ball the Bloch vectors of a fully mixed state and the pure states that form two bases, $\{|\uparrow\rangle, |\downarrow\rangle\}$, $\{|+\rangle, |-\rangle\}$.
  (ii) Find and diagonalise the states represented by Bloch vectors $\vec{r}_1 = (\frac{1}{2}, 0, 0)$ and $\vec{r}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

(b) Show that the operator ρ defined in Eq. (1) is a valid density operator for any vector $\vec{r}$ with $|\vec{r}| \leq 1$ by proving it fulfils the following properties:

  (i) Hermiticity: $\rho = \rho^\dagger$.
  (ii) Positivity: $\rho \geq 0$.
  (iii) Normalisation: $\mathrm{tr}(\rho) = 1$.

(c) Now do the converse: show that any qubit density operator may be written in the form of Eq. (1).

(d) Check that the surface of the ball is formed by all the pure states, i.e.

$$\rho \text{ pure } \Leftrightarrow \rho = \frac{1}{2}(\mathbb{I} + \vec{r} \cdot \vec{\sigma}) \text{ with } |\vec{r}| = 1. \tag{2}$$

(e) Using the Bloch sphere representation, find the Kraus operators fpr the depolarizing channel: $\rho \to \rho' = (1-p)\rho + p\frac{\mathbb{1}}{2}$.

Exercise 2. Distinguishing the states

Let us consider the following density operators of the bipartite system AB with $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where A and B are qubit systems:

$$\rho_{AB} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = |\psi_1\rangle\langle\psi_1| \quad \text{with} \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

$$\rho'_{AB} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\left(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}\right)$$

$$\rho''_{AB} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left(\frac{1}{2}\mathbb{I}_B\right) \otimes \left(\frac{1}{2}\mathbb{I}_B\right)$$

(a) Show that the reduced state of the system A is the same in all three examples (the same is true for the reduced state of the system B by virtue of symmetry):

$$\mathrm{tr}_B(\rho_{AB}) = \mathrm{tr}_B(\rho'_{AB}) = \mathrm{tr}_B(\rho''_{AB})$$

Suppose that Alice has access to the system A, and Bob has access to the system B. The fact the reduced states on the systems A and B are the same for all three examples means that Alice (or Bob) cannot distinguish the three situations through only local operations and measurements. However, there are global operations they can apply to distinguish the situations:

1. local measurements + classical communication, where they compare the outcomes and investigate correlations;
2. global measurements (harder to implement if Alice and Bob's labs are far apart).

(b) Show that for these measurements, the possible outcomes are as indicated on the table.

	$Z_A \otimes Z_B$	$X_A \otimes X_B$	a global measurement in Bell basis
ρ_{AB}	$(0,0)$ or $(1,1)$	$(+,+)$ or $(-,-)$	ψ_1
ρ'_{AB}	$(0,0)$ or $(1,1)$	$(+,+)$ or $(-,-)$ or $(+,-)$ or $(-,+)$	ψ_1 or ψ_2
ρ''_{AB}	$(0,0)$ or $(0,1)$ $(1,0)$ or $(1,1)$	$(+,+)$ or $(-,-)$ or $(+,-)$ or $(-,+)$	ψ_1 or ψ_2 or ψ_3 or ψ_4

where ψ_1, ψ_2, ψ_3 and ψ_4 denote the measurement results corresponding to the states in Bell basis:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right) \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} - |11\rangle_{AB}\right)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} + |10\rangle_{AB}\right) \qquad |\psi_4\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} - |10\rangle_{AB}\right)$$

Exercise 3. Kraus operators in the double slit experiment

Let us (once again) return to the double slit experiment. In Exercise Sheet 6, we looked at the interaction of of a particle going through a double slit and a measurement device (modelled as a qubit):

$$\underbrace{\left(\frac{1}{\sqrt{2}}|\psi_L\rangle_X + \frac{e^{i\alpha}}{\sqrt{2}}|\psi_R\rangle_X\right) \otimes |0\rangle_S}_{|\psi'\rangle_{XS}} \xrightarrow{U_{XS}} \underbrace{\frac{1}{\sqrt{2}}|\psi_L\rangle_X \otimes |0\rangle_S + \frac{e^{i\alpha}}{\sqrt{2}}|\psi_R\rangle_X \otimes |1\rangle_S}_{|\psi_0\rangle_{XS}}$$

In Exercise Sheet 8, we have also calculated reduced states of the systems X and S. We can consider the coupling as a quantum channel for the system X alone, which maps $\rho'_X = \mathrm{tr}_S\left(|\psi'\rangle\langle\psi'|_{XS}\right)$ to $\rho_X = \mathrm{tr}_S\left(|\psi_0\rangle\langle\psi_0|_{XS}\right)$.

(a) Find the Kraus operators for the coupling channel.

(b) What is the Stinespring dilation of the channel?