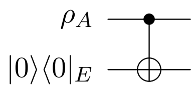
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Quantum Physics for Non-Physicists
Exercise Sheet 10

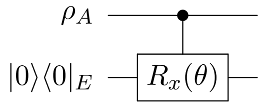
HS 2020
Dr. Lídia del Rio

Exercise 1. Controlled rotation gate: a weak measurement

In the lecture we have seen how to model a strong (projective) measurement as a CNOT gate between two qubits using one of them as a pointer:



Suppose now that instead of the CNOT gate we use a different controlled gate:



In this case, the operation on the system A alone corresponds to a channel $\rho_A \mapsto \rho'_A = \operatorname{tr}_E\left(U(\rho_A \otimes |0\rangle\langle 0|_E)U^\dagger\right)$ with $U = |0\rangle\langle 0|_A \otimes \mathbb{I}_E + |1\rangle\langle 1|_A \otimes R_x(\theta)$, where $R_x(\theta) = e^{-i\frac{\theta}{2}X} = \cos\frac{\theta}{2}\mathbb{I} - i\sin\frac{\theta}{2}X$ is the rotation about x axis in the Bloch sphere representation.

Hint: $R_x(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$

(a) Derive the states $\rho'_{AE} = U(\rho_A \otimes |0\rangle\langle 0|_E)U^\dagger$ and $\rho'_A = \operatorname{tr}_E\left(\rho'_{AE}\right)$ after applying the operation.

(b) Describe the final channel $\rho_A \mapsto \rho'_A$ as a function of θ. For which θ does it correspond to a strong measurement?

(c) For a given $R_x(\theta)$ find the corresponding POVM elements M_0 and M_1 associated with measurement outcomes 0 and 1.

Exercise 2. Measuring parity

In the lecture, we have seen how the parity observable looks like for a 2-qubit system AB:

$$\hat{P} = \Pi_0 - \Pi_1 = (|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}) - (|01\rangle\langle 01|_{AB} + |10\rangle\langle 10|_{AB})$$

Let us introduce a third qubit S initialized in the state $|0\rangle_S$. Design a unitary acting on the systems A, B and S which would perform a parity measurement on the pair AB and write down the result to the qubit S: the state of the system S would be flipped from $|0\rangle_S$ to $|1\rangle_S$ if the state of AB is odd, and stay the same if the state of AB is even.

Exercise 3. A system and a battery

In the lecture, we have seen how to make an energy exchange between a qubit (a two-level system) S and a battery B, with the state space characterized by $span\{|0\rangle_S, |1\rangle_S\} \otimes span\{|E_k\rangle_B\}_k$ and the evolution described by a Hamiltonian $H_{SB} = \epsilon|1\rangle\langle 1|_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes \sum_{k=-\infty}^{+\infty} \Delta k|E_k\rangle\langle E_k|$:

$$U_{SB} = |0\rangle\langle 1|_S \otimes \sum_k |E_{k+q}\rangle\langle E_k|_B + |1\rangle\langle 0|_S \otimes \sum_{k=-\infty}^{+\infty} |E_{k-q}\rangle\langle E_k|_B \quad \text{with} \quad q = \frac{\epsilon}{\Delta}$$

(a) Verify that U_{SB} is indeed a unitary operation: show that $U_{SB}U_{SB}^{\dagger} = \mathbb{I}$.

(b) Apply U_{SB} to the initial state $(\alpha|0\rangle_S + \beta|1\rangle_S)|E_k\rangle_B$. Compute the final state, and the final average energy in the system and in the battery.

Hint: For local Hamiltonians, the average energy of the system described by a density matrix ρ can be written as $\langle E \rangle = \mathrm{tr}(\rho H)$.

(c) Show that if $[U, H] = 0$, where H is the Hamiltonian of the system, and U is the applied unitary transformation, then the global average energy is conserved after applying U.

(d) Show that $[U_{SB}, H_{SB}] = 0$.