Given a system with Hamiltonian

$$H = \sum_i E_i |E_i\rangle\langle E_i|,$$

and a temperature T, we define the *thermal state*

$$\tau(T) = \frac{e^{-\frac{H}{kT}}}{Z},$$

where k is a constant (Boltzmann constant), and Z is the normalization factor which is called the *partition function*:

$$Z(T, H) = \sum_i e^{-\frac{E_i}{kT}}.$$

In this exercise sheet, we explore some properties of thermal states.

Exercise 1. Composability of thermal states

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two systems with the joint Hamiltonian

$$H_{AB} = H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B \quad \text{(the systems don't interact)}$$

(a) Show that in this case the thermal state of the joint system can be written as a tensor product of thermal states on individual subsystems:

$$\tau_{AB} = \tau_A \otimes \tau_B, \quad \text{or} \quad \frac{e^{-\frac{H_{AB}}{kT}}}{Z_{AB}} = \frac{e^{-\frac{H_A}{kT}}}{Z_A} \otimes \frac{e^{-\frac{H_B}{kT}}}{Z_B}$$

(b) Generalize the statement in (a) for the thermal state of n non-interacting subsystems.

Exercise 2. Energy and temperature

(a) For a qubit system with $H = \epsilon|1\rangle\langle 1|$, consider a thermal state $\tau(T)$ at a given temperature T. What is the relation between T and the average energy of the state $\langle E\rangle = \operatorname{tr}(H\tau(T))$?

(b) Now consider a set of n non-interacting identical qubits with the Hamiltonian

$$H_{\text{global}} = \epsilon \sum_{i=1}^{n} \mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_{i-1} \otimes |1\rangle\langle 1|_i \otimes \mathbb{I}_{i+1} \otimes \cdots \otimes \mathbb{I}_n$$

Express the global average energy $\langle E_{\text{global}}\rangle$ as a function of T.

(c) Now suppose that all we know is that the total energy of our n-qubit system ($n \gg 1$) is $\langle E_{\text{global}}\rangle = pn\epsilon$ with $0 < p < 1$ such that $pn \in \mathbb{N}$.

(i) If in addition we assume that the global state is thermal, what is the temperature T? What is the reduced state of the first qubit?

(ii) Suppose now that we don't want to assume that the global state is thermal. Instead, we know that it is a pure state $|\psi\rangle$ with energy $\langle E_{\text{global}}\rangle$, and (for simplicity) is diagonal in the energy basis (which is equivalent to saying that $[H, |\psi\rangle\langle\psi|] = 0$). The set of such states is called an *energy shell* for the energy $\langle E_{\text{global}}\rangle$:

$$\Omega_{\text{global}} := \{|\psi\rangle : \ [H, |\psi\rangle\langle\psi|] = 0 \text{ and } \operatorname{tr}(H|\psi\rangle\langle\psi|) = \langle E_{\text{global}}\rangle\}$$

Show that $|\Omega_{\text{global}}| = \binom{n}{pn}$, and that

$$\Omega_{\text{global}} = span\{|\psi_i\rangle\}_i, \quad |\psi_i\rangle = \sigma_i(|1\rangle^{\otimes np} \otimes |0\rangle^{\otimes n(1-p)}),$$

where σ_i is a permutation of subsystems.

(iii) Now we can represent our knowledge of the global state as a uniform distribution over all basis elements of Ω_{global}:

$$\rho_{\text{global}} = \frac{1}{|\Omega_{\text{global}}|} \sum_{|\psi_i\rangle} |\psi_i\rangle\langle\psi_i|$$

What is the reduced state of the first qubit?