Let us consider a system consisting of: a battery $\mathcal{H}_B$; a qubit $\mathcal{H}_S$; and a heat bath $\mathcal{H}_{\text{heatbath}}$. The heat bath, $\mathcal{H}_{\text{heatbath}} = \bigotimes_{\ell=1}^{N} \mathcal{H}_\ell$. The Hamiltonian of the heat bath assumes that the qubits do not interact:

$$H_{\text{heatbath}} = \sum_{\ell=1}^{N} H_\ell \otimes \mathbb{I}_{\text{rest}} = \sum_{\ell=1}^{N} \Delta \ell |1\rangle\langle 1|_\ell \otimes \mathbb{I}_{\text{rest}}$$

The battery is a system with an infinite number of equidistant energy levels (in a more realistic scenario, the number of levels should be finite, but here we will omit this detail for simplicity) with the Hamiltonian

$$H_B = \sum_{k=-\infty}^{+\infty} \Delta k |E_k\rangle\langle E_k|$$

As of the qubit $S$, both of its levels have the same energy, and $H_S = 0$. The global Hamiltonian is expressed as the sum of individual Hamiltonians:

$$H_{\text{global}} = H_B + H_S + H_{\text{heatbath}} = H_B + H_{\text{heatbath}}$$

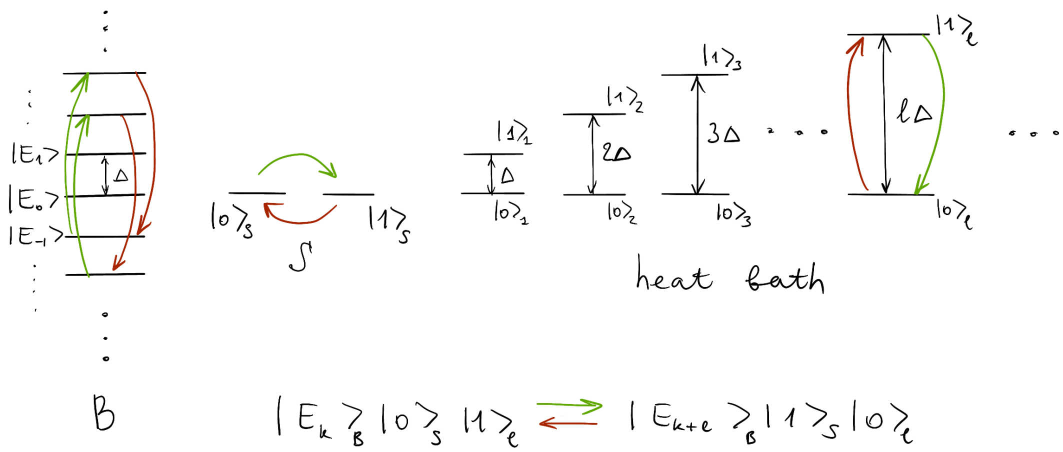


Figure 1: The unitary process of energy exchange on three systems.

Now imagine that we consequently apply $N$ unitaries to the joint system, which would allow us to transfer the energy from the $l$th qubit of heat bath to the battery (and back) by flipping the state of the qubit in the process (Figure 1):

$$U = U^{(N)} \dots U^{(2)} U^{(1)}, \text{ where}$$

$$U^{(\ell)} = \sum_{k=-\infty}^{+\infty} [|E_{k+\ell}\rangle_B |1\rangle_S |0\rangle_\ell \langle E_k|_B \langle 0|_S \langle 1|_\ell + |E_k\rangle_B |0\rangle_S |1\rangle_\ell \langle E_{k+\ell}|_B \langle 1|_S \langle 0|_\ell] \otimes \mathbb{I}_{\text{rest}}$$
$$+ \mathbb{I}_B \otimes [|0\rangle_S |0\rangle_\ell \langle 0|_S \langle 0|_\ell + |1\rangle_S |1\rangle_\ell \langle 1|_S \langle 1|_\ell] \otimes \mathbb{I}_{\text{rest}}$$

(a) Show that the total energy of the joint system is preserved in the process:

$$[U, H_{\text{global}}] = 0$$

*Hint: Show that* $[U^{(\ell)}, H_{global}] = 0\ \forall l$.

(b) Suppose that initially the energy of the battery is $E_0$, the qubit is in a maximally mixed state, and the heat bath is a thermal state at temperature $T$ (here for simplicity we assume the Boltzmann constant $k_B = 1$):

$$\rho^{(0)} = |E_0\rangle\langle E_0|_B \otimes \frac{1}{2}\mathbb{I}_S \otimes \tau_{\text{heatbath}}(T) = |E_0\rangle\langle E_0|_B \otimes \frac{1}{2}\mathbb{I}_S \otimes \bigotimes_{\ell=1}^{N} \frac{1}{Z(H_\ell, T)}(|0\rangle\langle 0|_\ell + e^{-\frac{l\Delta}{T}}|1\rangle\langle 1|_\ell)$$

(i) Compute the state of the system after the first unitary is applied: $\rho^{(1)} = U^{(1)}\rho^{(0)}U^{(1)\dagger}$.

(ii) Compute the reduced states of the systems involved in the first transformation $\rho^{(1)}_{BS}$, $\rho^{(1)}_B$, $\rho^{(1)}_S$, $\rho^{(1)}_{\ell=1}$. How are the states of the battery $B$ and the qubit $S$ now correlated? How can you represent the process as a random walk for systems $B$ and $S$?

(iii) Calculate the energy difference of the battery $\delta E^{(1)}_B = \mathrm{tr}(H_B\rho^{(1)}_B) - \mathrm{tr}(H_B\rho^{(0)}_B)$, the thermal qubit $\delta E_1$ and the qubit $\delta E^{(1)}_S$.

(c) Let us now extrapolate what happens after we apply the remaining $N-1$ unitaries (and let us not think about the final state of all three systems at once, that might lead to a headache).

(i) Argue (e.g. by induction) that after $k$th step, the reduced state of the qubit $S$ can written as

$$\rho^{(k)}_S = \frac{e^{-\frac{k\Delta}{T}}}{(1+e^{-\frac{k\Delta}{T}})}|1\rangle\langle 1| + \frac{1}{(1+e^{-\frac{k\Delta}{T}})}|0\rangle\langle 0|,$$

and the reduced state of the used thermal qubit is fully mixed.

(ii) What is the net change in the energy of the battery in $N$ steps? Program the random walk numerically for $N = 20$ steps. Calculate the energy variance.