Exercise 1. Uncertainty relation

(a) Prove the Schwarz inequality for two vectors $|a\rangle, |b\rangle \in \mathcal{H}$:

$$\langle a|a\rangle\langle b|b\rangle \geq |\langle a|b\rangle|^2$$

Hint: One way to prove the inequality is to decompose the vectors in the same orthonormal basis.

(b) Consider a qubit in state

$$|\psi\rangle = e^{i\phi}\cos\theta|0\rangle + e^{-i\phi}\sin\theta|1\rangle$$

For the observables Z, Y and X find ΔX, ΔY and $\langle Z\rangle$. Show that $\Delta X\Delta Y \geq \langle \frac{1}{2}Z\rangle$.

(c) Prove the uncertainty relation for the mixed states: given operators A and B, show that for all states ρ

$$\langle \Delta A\rangle_\rho \langle \Delta B\rangle_\rho \geq \frac{1}{2}|\langle [A,B]\rangle_\rho|$$

Exercise 2. Spin-1

For a spin-1 system, compute the matrix form of S_z, S_+, S_- and S^2 in the basis $\{|-1\rangle, |0\rangle, |1\rangle\}$ where S_z is diagonal.

Hint: Start with S_z: use that it has three eigenvalues $m = -\hbar, 0, \hbar$. Then to derive S_+ and S_-, apply the formulas obtained in the lecture:

$$S_+ = \sum_m \hbar\sqrt{s(s+1) - m(m+1)}|m+1\rangle\langle m|$$
$$S_- = \sum_m \hbar\sqrt{s(s+1) - m(m-1)}|m-1\rangle\langle m|$$

Exercise 3. Harmonic oscillator

In this exercise, we will look at a system, which we will refer to as *harmonic oscillator*, described by a Hilbert space $\mathcal{H}$ of infinite dimension. This system acts similarly to a spin system: X and P play the part of S_x and S_y, and we will define ladder operators analogously to S_- and S_+. The energy will be analogous to S_z. To see this, let us consider the Hamiltonian of the harmonic oscillator:

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

Denote the eigenstates of the Hamiltonian $|n\rangle \in \mathcal{H}$, where $H|n\rangle = E_n|n\rangle$. Let us introduce ladder operators a (lowering) and $a^\dagger$ (raising):

$$a = \sqrt{\frac{m\omega}{2\hbar}}X + \frac{i}{\sqrt{2m\omega\hbar}}P$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - \frac{i}{\sqrt{2m\omega\hbar}}P$$

(a) Show that $[a, a^\dagger] = 1$.

Hint: Use the commutation relation $[X, P] = i\hbar$.

(b) Rewrite the Hamiltonian as $H = \hbar\omega(a^\dagger a + aa^\dagger) = \hbar\omega(a^\dagger a + \frac{1}{2})$.

(c) Compute $[H, a]$ and $[H, a^\dagger]$. Use them to show

$$H(a|n\rangle) = (E_n - \hbar\omega)(a|n\rangle)$$
$$H(a^\dagger|n\rangle) = (E_n + \hbar\omega)(a^\dagger|n\rangle)$$

(d) Show that we can summarize the results above by denoting the states with energy $E_n \pm \hbar\omega$ as $|n \pm 1\rangle$:

$$a|n\rangle = c_n|n-1\rangle \quad a^\dagger|n\rangle = d_n|n+1\rangle$$

for some constants c_n, d_n. Determine these constants from the normalization condition for states $|n\rangle$.

(e) Show that the eigenvalues have the form $E_n = \hbar\omega(n + \frac{1}{2})$ for $n \in \mathbb{N}_0$.

(f) *Extra*: draw the wave functions of the first 3 energy eigenstates.