Exercise 1. *Practicing Dirac notation: finite Hilbert spaces*

Consider a three-dimensional Hilbert space with an orthonormal basis $|1\rangle, |2\rangle, |3\rangle$. Using $a, b, c \in \mathbb{C}$, let us define the states:

$$|\psi\rangle = a|1\rangle - b|2\rangle + c|3\rangle, \qquad |\phi\rangle = b|1\rangle + a|2\rangle.$$

(a) *Write down $\langle\phi|$ and $\langle\psi|$. Check that $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$.*

Solution We can write $\langle\phi|$ and $\langle\psi|$ as

$$\langle\psi| = a^*\langle 1| - b^*\langle 2| + c^*\langle 3|, \qquad \langle\psi| = b^*\langle 1| + a^*\langle 2|.$$

Using the fact that the basis $|1\rangle, |2\rangle, |3\rangle$ is orthonormal, we arrive to:

$$\langle\phi|\psi\rangle = (b^*\langle 1| + a^*\langle 2|)(a|1\rangle - b|2\rangle + c|3\rangle) = b^*a - a^*b;$$
$$\langle\psi|\phi\rangle = (a^*\langle 1| - b^*\langle 2| + c^*\langle 3|)(b|1\rangle + a|2\rangle) = a^*b - b^*a = (b^*a - a^*b)^* = \langle\phi|\psi\rangle^*.$$

(b) *What conditions should a, b and c satisfy such that both states are normalized, i.e. $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1$?*

Solution The normalization identity imposes the following constraints:

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2 + |c|^2 = 1;$$
$$\langle\phi|\phi\rangle = |b|^2 + |a|^2 = 1.$$

It follows that $|a|^2 + |b|^2 = 1$ and $c = 0$.

(c) *Express $|\phi\rangle$ and $|\psi\rangle$ as column vectors. Calculate their inner product.*

Solution In the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis, the states can be expressed as vectors:

$$|\psi\rangle = \begin{pmatrix} a \\ -b \\ 0 \end{pmatrix}, \qquad |\phi\rangle = \begin{pmatrix} b \\ a \\ 0 \end{pmatrix}.$$

Their inner product is $\langle\psi|\phi\rangle = \begin{pmatrix} a^* & -b^* & 0 \end{pmatrix} \begin{pmatrix} b \\ a \\ 0 \end{pmatrix} = a^*b - b^*a.$

(d) *Let $Q = |\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|$. Is Q hermitian? What are its eigenvalues?*

Solution Q is hermitian as $Q^\dagger = Q$. In order to find its eigenvalues, let us express it in $|1\rangle, |2\rangle, |3\rangle$ basis:

$$Q = \begin{pmatrix} a \\ -b \\ 0 \end{pmatrix} \begin{pmatrix} a^* & -b^* & 0 \end{pmatrix} + \begin{pmatrix} b \\ a \\ 0 \end{pmatrix} \begin{pmatrix} b^* & a^* & 0 \end{pmatrix} = \begin{pmatrix} |a|^2 & -b^*a & 0 \\ -ba^* & |b|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} |b|^2 & ba^* & 0 \\ ab^* & |a|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & ba^* - b^*a & 0 \\ ab^* - ba^* & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let us denote $\alpha = ba^* - b^*a$. Then the eigenvalues of Q are $\{1 - |\alpha|, 1 + |\alpha|, 0\}$.
Comment. A simple way to see that Q has a zero eigenvalue is to note that $Q|3\rangle = 0$.

Exercise 2. ***A particle in a box: continuous Hilbert spaces***

Let $\mathcal{H}$ be an infinite dimensional (continuous) Hilbert space, and let $\{|x\rangle\}_{x\in\mathbb{R}}$ and $\{|p\rangle\}_{p\in\mathbb{R}}$ be two orthonormal bases. We can expand any state $|\psi\rangle$ in one of these bases, e.g. $|\psi\rangle = \int_{-\infty}^{\infty} dx\psi(x)|x\rangle$. We call the function $\psi(x)$ the position wave function of the state. Here, we are interested in "conjugate bases", i.e. two bases related by a Fourier transform,

$$|p\rangle = \beta \int_{-\infty}^{\infty} e^{-i\alpha px}|x\rangle dx$$

For reasons that we will see later, we call $\{|x\rangle\}_{x\in\mathbb{R}}$ the position basis, and $\{|p\rangle\}_{p\in\mathbb{R}}$ the momentum basis.

(a) *Find the coefficients α, β which normalize the state.*

Solution The normalization condition $\langle p|q\rangle = \delta(p-q)$ leaves us with:

$$\langle p|q\rangle = |\beta|^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\alpha qx + i\alpha py}\langle y|x\rangle dxdy = |\beta|^2 \int_{\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\alpha(qx-py)}\delta(x-y)dxdy =$$
$$|\beta|^2 \int_{-\infty}^{\infty} e^{-i\alpha x(q-p)}dx = |\beta|^2\frac{2\pi}{\alpha}\delta(q-p) \quad\Rightarrow\quad |\beta|^2\frac{2\pi}{\alpha} = 1$$

Canonically, we choose $\alpha = 1$ and $\beta = \frac{1}{\sqrt{2\pi}}$.

(b) *Show that the position and the momentum bases are mutually unbiased, that is,*

$$|\langle x|p\rangle|^2 = |\langle x'|p'\rangle|^2 \qquad \forall x, x', p, p' \in \mathbb{Z}.$$

Solution Let us consider arbitrary $x, p \in \mathbb{Z}$. Then

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dye^{-ipy}\langle x|y\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dye^{-ipy}\delta(x-y) = \frac{1}{\sqrt{2\pi}}e^{-ip'x'} \quad\Rightarrow\quad |\langle x|p\rangle|^2 = \frac{1}{2\pi}.$$

As a more concrete example, let us consider a quantum particle confined in a box of length L with impenetrable walls, the centre of the wall being at the origin of the coordinates. In the ground state, the wave function of the particle is given by [1]:

$$\psi_1(x) = \begin{cases} \gamma\cos(\frac{\pi x}{L}) & \textit{if } -\frac{L}{2} < x < \frac{L}{2} \\ 0 & \textit{otherwise} \end{cases}$$

[1] the exact derivation will be given later in the course.

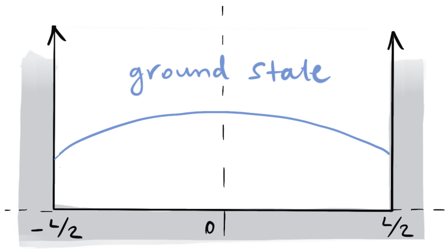

Figure 1: The ground state position wavefunction of a particle in a box.

(c) Normalize the wave function to determine the coefficient γ.

Solution The wave function in Dirac notation: $|\psi_1\rangle = \int \psi_1(x)|x\rangle dx$. The normalization condition $\langle\psi_1|\psi_1\rangle = 1$ gives:

$$\langle\psi_1|\psi_1\rangle = \int\int \psi_1(x)\psi_1^*(y)\langle y|x\rangle dxdy = \int\int \psi_1(x)\psi_1^*(y)\delta(x-y)dxdy = \int |\psi_1(x)|^2 dx$$
$$= |\gamma|^2 \int_{-L/2}^{L/2} \cos^2(\frac{\pi x}{L})dx = |\gamma|^2 \frac{L}{2\pi}\int_{-\pi}^{\pi} \frac{1+\cos x}{2}dx = \frac{L}{2}|\gamma|^2 = 1$$

Hence, up to a phase factor, $\gamma = \sqrt{\frac{2}{L}}$.

(d) Write down the wave function in the momentum basis.

Solution To express the position basis elements via the momentum basis, we use the inverse Fourier transform:

$$|\psi_1\rangle = \int \psi_1(x)|x\rangle dx = \int \psi_1(x)\frac{1}{\sqrt{2}}\int e^{ipx}|p\rangle dpdx = \int \left(\frac{1}{\sqrt{2}}\int \psi_1(x)e^{ipx}dx\right)|p\rangle dp$$

It follows that the momentum wave function can be expressed as:

$$\tilde{\psi_1}(p) = \frac{1}{\sqrt{2}}\int \psi_1(x)e^{ipx}dx = \frac{1}{\sqrt{L}}\int_{-L/2}^{L/2}\cos(\frac{\pi x}{L})e^{ipx}dx = \frac{\pi\sqrt{L}}{\pi^2 - p^2L^2}\cos\frac{pL}{2}.$$

(e) Suppose that we want to perform a measurement of the position of of the particle. What is the expected value of the position x? What is the expected value of the momentum p if we choose to measure it instead?

Solution The expected value of the position is given by (the integrated function is odd):

$$\langle x\rangle = \langle\psi_1|\hat{x}|\psi_1\rangle = \int_{-L/2}^{L/2} |\psi_1(x)|^2 x dx = \frac{2}{L}\int_{-L/2}^{L/2} x\cos^2(\frac{\pi x}{L})dx = 0$$

It is most likely to find the particle in the middle of the box. Analogously, working in the momentum basis we obtain:

$$\langle p\rangle = \int_{-\infty}^{\infty} |\tilde{\psi_1}(p)|^2 p dp = \int_{-\infty}^{\infty} \frac{\pi^2 L}{(\pi^2 - p^2L^2)^2} p\cos^2\frac{pL}{2}dp = 0.$$

Exercise 3. ***Bloch sphere and measurements***

In this exercise you will be introduced to basic single-qubit gates. The basic gates you will need are (given in matrix form in the computational Z basis):

- *Pauli-X, Pauli-Y and Pauli-Z:*

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{1}$$

- *Rotation around the x/y/z - axis by an angle θ:*

$$R_x(\theta) = e^{-i\theta X/2} = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}, \quad R_y(\theta) = e^{-i\theta Y/2} = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix},$$
$$R_z(\theta) = e^{-i\theta Z/2} = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}, \tag{2}$$

You may already know that any pure state of a qubit can be described and visualised as a point on a 3D ball of radius 1: this is the so-called Bloch sphere. In this question we will familiarise ourselves with this representation.

(a) *Show that any pure state $|\psi\rangle$ of a single qubit can be written as $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. These are the polar coordinates of state $|\psi\rangle$ on the Bloch sphere.*

Solution Any single-qubit state can be written as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. Separating the magnitude and phase of each complex number, the state can be written as:

$$|\psi\rangle = |\alpha|e^{i\mathrm{Arg}(\alpha)}|0\rangle + |\beta|e^{i\mathrm{Arg}(\beta)}|1\rangle. \tag{S.1}$$

Since all states which differ only by the global phase are equivalent [2], we can re-write the state as:

$$|\psi\rangle = |\alpha||0\rangle + |\beta|e^{i\phi}|1\rangle, \tag{S.2}$$

with

$$\phi = (\mathrm{Arg}(\beta) - \mathrm{Arg}(\beta)) \mod 2\pi \in [0, 2\pi]. \tag{S.3}$$

Finally, any two positive numbers $|\alpha|, |\beta|$ that satisfy $|\alpha|^2 + |\beta|^2 = 1$ can be written as $|\alpha| = \cos(\theta/2), |\beta| = \sin(\theta/2)$ with $\theta \in [0, \pi]$. Hence the state can indeed be written as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle. \tag{S.4}$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$

(b) *Write down the the (θ, ϕ) coordinates for each of the following states. Also draw all the state vectors on the Bloch sphere (a state vector is a vector from the centre of the co-ordinate system to the point representing the state).*

$$|0\rangle \quad ; \quad |1\rangle \quad ; \quad |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \quad ; \quad |\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle) \; .$$

These states are the eigenstates of Pauli Z, X and Y operators, respectively.

[2] More about this later in the semester.

Solution The association goes as follows:

$$
\begin{aligned}
|0\rangle &: \theta = 0, \phi = \text{anything} \\
|1\rangle &: \theta = \pi, \phi = \text{anything} \\
|+\rangle &: \theta = \frac{\pi}{2}, \phi = 0 \\
|-\rangle &: \theta = \frac{\pi}{2}, \phi = \pi \\
|+i\rangle &: \theta = \frac{\pi}{2}, \phi = \frac{\pi}{2} \\
|-i\rangle &: \theta = \frac{\pi}{2}, \phi = -\frac{\pi}{2}
\end{aligned}
$$

These are visualised in Fig. 2.

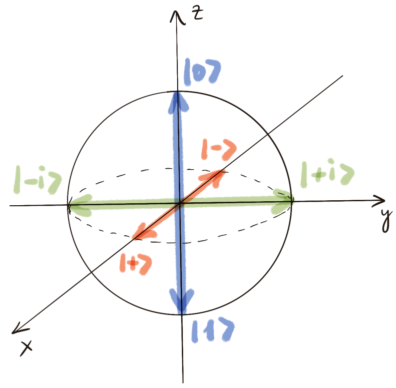

Figure 2: Bloch sphere with state vectors.

(c) Show that implementing $R_z(\alpha)$ (as given in (2)) corresponds to a rotation by the angle α around the z-axis.

Solution Let us consider an arbitrary pure state $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$. On the Bloch sphere this state has the angle θ to the z-axis and the angle ϕ in the x-y-plane to the x-direction. If we apply $R_z(\alpha)$, the final state is given by

$$|\psi'\rangle = R_z(\alpha)|\psi\rangle = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = e^{-i\alpha/2} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i(\phi+\alpha)}\sin\frac{\theta}{2} \end{pmatrix}. \qquad \text{(S.5)}$$

The global phase $e^{-i\alpha/2}$ does not matter (and can not be seen on the Bloch sphere). On the Bloch sphere, the final state has still the angle θ to the z-axis, but has now an angle $\phi + \alpha$ in the x-y-plane to the x-direction. Thus, the state was rotated by the angle α around the z-axis.

(d) Let us now discuss measurements. Recall that any Hermitian operator M can correspond to a quantum measurement — the measurement basis is the eigenbasis of M. We perform a Z measurement on a state with Bloch coordinates (θ, ϕ). This is called the computational basis*. What are the possible outcomes, their probabilities and possible post-measurement states? And for an X measurement?*

Solution The probability to measure outcome 1 is $|\langle 0|\psi\rangle|^2 = \cos^2(\theta/2)$. The post-measurement state is then $|0\rangle$. The probability to measure outcome -1 is $|\langle 1|\psi\rangle|^2 = \sin^2(\theta/2)$ and the corresponding post-measurement state is $|1\rangle$.
The eigenstates of the X measurement are $|\pm\rangle$ with eigenvalues ± 1 respectively. So the probability of obtaining outcome $+1$ is:

$$P(+1) = |\langle +|\psi\rangle|^2 = \frac{1}{2}|\cos(\theta/2) + e^{i\phi}\sin(\theta/2)|^2 = \frac{1}{2}(1 + \sin(\theta)\cos(\phi)) \tag{S.6}$$

and the corresponding post-measurement state is $|+\rangle$. The other probability is $P(-1) = 1 - P(1)$ and the post-measurement state is $|-\rangle$.

(e) *A qubit synthesiser in a quantum factory is supposed to reliably produce states* $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. *However, an unconfirmed report suggests the machine fails to produce* $|+\rangle$. *Instead, it randomly produces either* $|0\rangle$ *(50% of the time) or* $|1\rangle$ *(also 50% of the time). A quantum mechanic is sent to investigate. He is allowed to perform measurements on the outcoming qubits. How shall the quantum mechanic debug the machine?*

Solution For the state $|+\rangle$ the probability to measure $+1$ in the Z basis is $\frac{1}{2}$. This is the same as if the state was prepared randomly in either $|0\rangle$ or $|1\rangle$ and hence the Z measurement cannot help us debug the machine.
On the either hand, if the state is $|+\rangle$, then the X measurement will always return $+1$. Analysing further, we see that an X measurement of $|0\rangle$ returns $+1$ only 50% of the time, and an X measurement of $|1\rangle$ also returns $+1$ only 50% of the time. This means that the output of a faulty machine, when measured in X basis, will return -1 50% of the time. A mechanic can therefore perform repeated measurements with the X-measurement ruler. If the output is -1 at least once, then the machine is for sure faulty. If however N measurements return only $+1$, then the machine works as desired with probability $P = 1 - \frac{1}{2^N}$.