Exercise 1. *Unitary and Hermitian operators*

Let $\mathcal{H}$ be a Hilbert space and $A, B \in End(\mathcal{H}, \mathcal{H})$ operators in that Hilbert space. Here you have to prove some of their properties.
Note: the operator exponential is given by the power series:

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

(a) *Show that $(e^A)^\dagger = e^{A^\dagger}$.*

Solution

$$(e^A)^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} (A^n)^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} (A^\dagger)^n = e^{A^\dagger}.$$

(b) *Suppose that $[A, B] = AB - BA = 0$, that is, the operators A and B commute. Prove that $e^{A+B} = e^A e^B$.*

Solution The operators A and B commute, hence $AB = BA$ – which means that the order does not matter for these operators when they are multiplied, and we can use the binomial theorem in a usual way:

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{1}{n!} (A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} C_n^m A^m B^{n-m} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} A^m B^{n-m}$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{m!(n-m)!} A^m B^{n-m} = e^A e^B.$$

In the last step, we have used the Cauchy formula for the product of two series.

(c) *Show that if the operator A is Hermitian ($A = A^\dagger$) , then $U = e^{iA}$ is unitary ($UU^\dagger = U^\dagger U = \mathbb{I}$). Show also that for a collection $\{A_j\}_j$ of Hermitian operators , $U = \bigotimes_j e^{iA_j}$ is unitary.*

Hint: Make use of the results in (a) and (b).

Solution From (a) it follows that $U^\dagger = e^{-iA^\dagger} = e^{-iA}$. Then since $[A, A] = 0$ (every operator commutes with itself),

$$U^\dagger U = e^{-iA} e^{iA} = e^{-iA+iA} = \mathbb{I}.$$

(d) *Show that if U is a unitary, then there exists a Hermitian operator A such that $U = e^{iA}$.*

Solution Let us write U in its diagonal form: $U = WDW^\dagger$ where $D = diag(e^{i\alpha_1}, e^{i\alpha_2}, \dots)$ with $\alpha_j \in \mathbb{R}$, and W is a unitary. Let us choose $H = Wdiag(\alpha_1, \alpha_2, \dots)W^\dagger$. H is Hermitian, as α_j are real, and

$$e^{iH} = \sum_{n=0}^{\infty} \frac{1}{n!}(Wdiag(\alpha_1, \alpha_2, \dots)W^\dagger)^n = \sum_{n=0}^{\infty} \frac{i^n}{n!} Wdiag(\alpha_1, \alpha_2, \dots)^n W^\dagger$$
$$= We^{i \cdot diag(\alpha_1, \alpha_2, \dots)}W^\dagger = WDW^\dagger = U.$$

(e) *Suppose that V is both unitary and Hermitian. Show that the only possible eigenvalues for V are ± 1 and that $V^2 = \mathbb{I}$.*

Solution From the definitions of Hermitian and unitary operators it follows that $\mathbb{I} = V^\dagger V = VV = V^2$. Suppose that $|\phi\rangle$ is an eigenvector of V, corresponding to the eigenvalue λ: $V|\phi\rangle = \lambda|\phi\rangle$. The complex conjugate reads $\langle\phi|V^\dagger = \langle\phi|\lambda^*$; the product of these two expressions gives

$$\lambda\lambda^* = \langle\phi|V^\dagger V|\phi\rangle = 1 \Rightarrow |\lambda|^2 = 1.$$

Additionally, note

$$\lambda^* = \langle\phi|V^\dagger|\phi\rangle = \langle\phi|V|\phi\rangle = \lambda$$

Hence, λ is real, and $|\lambda|^2 = 1$, which means $\lambda = \pm 1$.

(f) *Suppose that an observable $A = A(t)$ is time-dependent, and H is the Hamiltonian of the system. Show that*

$$\frac{d}{dt}\langle A\rangle = \frac{1}{i\hbar}\langle\psi|[A, H]|\psi\rangle + \langle\psi|\frac{dA}{dt}|\psi\rangle$$

Solution From Schrödinger's equation it follows that $i|\psi\rangle = H|\psi\rangle$ (we assume $\hbar = 1$). The conjugate transpose of gives us the evolution of the bra: $-i\langle\psi| = \langle\psi|H^\dagger = \langle\psi|H$ (the Hamiltonian H is Hermitian). Then we can take the time derivative by parts:

$$\frac{d}{dt}\langle A\rangle = \frac{d}{dt}(\langle\psi|A|\psi\rangle) = (\frac{d}{dt}\langle\psi|)A|\psi\rangle + \langle\psi|(\frac{dA}{dt})|\psi\rangle + \langle\psi|A(\frac{d}{dt}|\psi\rangle)$$
$$= i\langle\psi|HA|\psi\rangle + \langle\psi|(\frac{dA}{dt})|\psi\rangle - i\langle\psi|AH|\psi\rangle = -i\langle\psi|[A, H]|\psi\rangle + \langle\psi|\frac{dA}{dt}|\psi\rangle.$$

Exercise 2. *Elitzur-Vaidman bomb test*

Suppose that you have a box that is either empty or contains a very sensitive bomb that would explode if hit by only a single photon of light. You have to determine whether the box is empty or not (while staying alive). This can be done with high probability using a trick known as interaction-free measurement.

Let us model a photon as a two-level quantum system, e.g. a qubit. To carry out the test, we need a Mach-Zehnder interferometer like the one we saw last week. The difference is that now we use beam splitters that modify the state of the photon as given by the unitary $U(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ in the computational basis $\{|0\rangle, |1\rangle\}$ ($|0\rangle$ and $|1\rangle$ can be understood as corresponding to transmitted and reflected

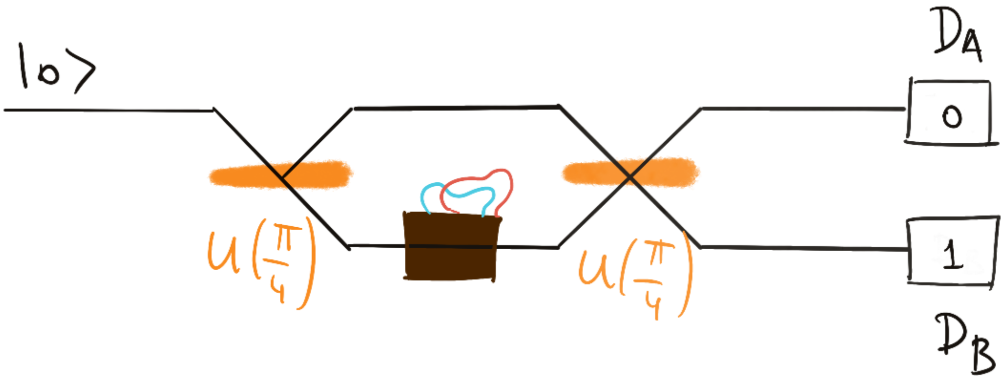


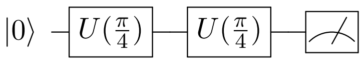
Figure 1: The experimental setup of Elitzur-Vaidman bomb tester.

beams). To test whether the box contains a bomb, you can make two tiny holes at its opposite sides such that only a single photon could pass through, and place it on one of the two paths of the interferometer, say the bottom one.

The photon starts out in the state $|0\rangle$. It then passes through a $\phi = \pi/4$ beam splitter which sends it onto two possible paths in superposition. The bottom path goes through the box, potentially resulting in a measurement due to the bomb. The two paths are then recombined by another $\phi = \pi/4$ beam splitter, and the photon is measured at the end by the detectors D_A, D_B in the $\{|0\rangle, |1\rangle\}$ basis by a projective measurement.

(a) *Assuming there was no bomb inside the box, derive the state before the final measurement.*

Solution To see the evolution of the state, it might be useful to draw the quantum circuit of the experiment (assuming there is no bomb, hence no measurement in between):



The final state just before the measurement can be written as:

$$|\psi_{final}^{no\ bomb}\rangle = U(\frac{\pi}{4})U(\frac{\pi}{4})|0\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = |1\rangle$$

We conclude that the final measurement would yield 1 with probability 100%.

(b) *Assuming there was a bomb inside the box, what is the probability of triggering the bomb?*

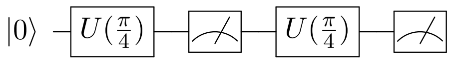
Solution To answer this question, we need to look at the state of the photon just after exiting the first beam splitter:

$$|\psi_1\rangle = U(\frac{\pi}{4})|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

The probability of the photon being measured on the bottom path (in the state $|1\rangle$) is then equal to 50% (already a better chance of survival, compared to the classical case!).

(c) *Assuming there was a bomb and it did not explode, derive the state before the final measurement.*

Solution If there was a bomb in the box, a measurement happens in between two beam splitters, and the circuit takes the form (keep in mind that the whole circuit up to the final measurement is no longer unitary):



The bomb not exploding means that the first measurement resulted in the photon collapsing into $|0\rangle$ state. After this, it went through the second beam splitter, which gives us the state before the final measurement:

$$|\psi_{final}^{bomb}\rangle = U(\frac{\pi}{4})|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

In the case the bomb is there, and it does not explode, the final state looks like the photon only passed one beam splitter instead of two.

(d) *Assume you run the experiment without knowing if the bomb is present. If no explosion occurred, what can you conclude from the outcome of the final measurement? What do you learn if the outcome was 1 and what do you learn if the outcome was 0?*

Solution Summarizing our results above, we can state the following:

- if the bomb is not present, the final measurement yields 1 with 100% probability;
- if the bomb is present, we survive with 50% probability, and in that case the final measurement yields 1 and 0 with equal probability.

Hence, if we get the outcome 0, we can with certainty state that the bomb is present. If we get the outcome 1, the box may be empty – or it may not.

Suppose that the box contains the bomb with 50% probability. Then you stay alive with probability 75%, and obtain the outcome 1 with probability 5/6, and 0 with probability 1/6. If the outcome is 0, then we can make a guess (a bomb is in the box) with 100% probability; if it is 1, our guess (the box is empty) is correct with probability 3/4. All in all, our guess is correct with $1/6 \cdot 1 + 5/6 \cdot 3/4 = 19/24$ probability.

Now let the initial state be $|0\rangle$ as before. For some integer $n \geq 1$ repeat the following two steps n times: apply the beam splitter $U(\pi/(2n))$ and let the the bottom path go through the box.

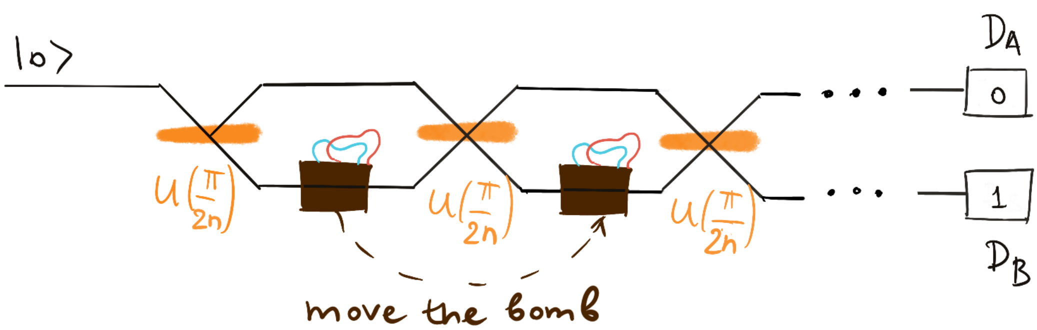
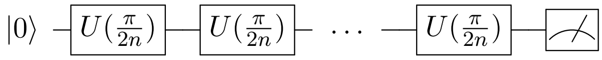


Figure 2: The experimental setup of Elitzur-Vaidman bomb tester with n copies of beam splitter (after each step we move the box further along the interferometer).

(e) *What is the final state if there was no bomb?*

Solution Again, let us represent the setting as a quantum circuit:



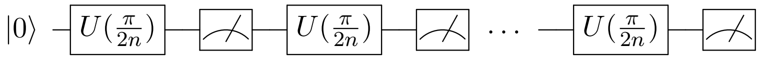
The final state just before the measurement:

$$|\Psi_{final}^{no\ bomb}\rangle = U^n|0\rangle = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^n |0\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle$$

This result is similar to the case of $n = 2$ discussed above.

(f) *What is the final state if there was a bomb and it did not explode in any of the n trials?*

Solution The circuit for this case reads:



The bomb did not explode, hence all measurements in between collapsed the photon into the $|0\rangle$ state (including the measurement just before the last beam splitter). The final state before the measurement:

$$|\Psi_{final}^{no\ bomb}\rangle = U(\frac{\pi}{2n})|0\rangle = \cos\frac{\pi}{2n}|0\rangle + \sin\frac{\pi}{2n}|1\rangle$$

(g) *Assuming the bomb is present, compute the probability of you still being alive at the end of the full test as a function of n. Additionally, compute the probability of guessing correctly (conditioned on you being alive) after the test. What value of n should you choose so that you are still alive with probability 99.9%? And to guess correctly with probability 95%?*

Solution Summarizing information gathered above,

- if the bomb is not present, the final measurement yields 1 with 100% probability;
- if the bomb is present, we survive with $\cos^2\frac{\pi}{2n}$ probability, and in that case the final measurement yields 1 with probability $\sin^2\frac{\pi}{2n}$ and 0 with probability $\cos^2\frac{\pi}{2n}$.

The probability of survival (if the bomb is present) is hence $P_{alive} = \cos^2\frac{\pi}{2n}$. For $P_{alive} = 0.999$, we need $n \approx 110$ beam splitters.

Suppose that the box contains the bomb with 1/2 probability. Then you stay alive with probability $\frac{1}{2} + \frac{1}{2}\cos^2\frac{\pi}{2n}$, and obtain the outcome 1 with probability

$$\frac{\frac{1}{2} + \frac{1}{2}\cos^2\frac{\pi}{2n}\sin^2\frac{\pi}{2n}}{\frac{1}{2} + \frac{1}{2}\cos^2\frac{\pi}{2n}} = \frac{1 + \cos^2\frac{\pi}{2n}\sin^2\frac{\pi}{2n}}{1 + \cos^2\frac{\pi}{2n}},$$

and 0 with probability

$$\frac{\frac{1}{2}\cos^2\frac{\pi}{2n}\cos^2\frac{\pi}{2n}}{\frac{1}{2} + \frac{1}{2}\cos^2\frac{\pi}{2n}} = \frac{\cos^4(\frac{\pi}{2n})}{1 + \cos^2\frac{\pi}{2n}}.$$

If the outcome is 0, then we can make a guess (a bomb is in the box) with 100% probability; if it is 1, our guess (the box is empty) is correct with probability $\frac{1}{\sin^2\frac{\pi}{2n}+1}$. All in all, our guess is correct with probability

$$\frac{1 + \cos^2\frac{\pi}{2n}\sin^2\frac{\pi}{2n}}{1 + \cos^2\frac{\pi}{2n}} \cdot \frac{1}{\sin^2\frac{\pi}{2n} + 1} + \frac{\cos^4(\frac{\pi}{2n})}{1 + \cos^2\frac{\pi}{2n}}$$
$$\approx 1 - \frac{1}{2}(\frac{\pi}{2n})^2 + O((\frac{\pi}{2n})^4) \text{ for big } n$$

To guess correctly with probability 95% (in the case we stay alive), we need $n \approx 5$ beam splitters.