## Exercise 1. *Wave function in 3D*

*In the lecture, we have seen how a wave function can be introduced in the three-dimensional space $\psi(\vec{r})$.*

(a) *Show that the momentum wave function in 3D can be written as*

$$\bar{\psi}(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint d\vec{r} e^{-i\vec{p}\cdot\vec{r}/\hbar}\psi(\vec{r})$$

**Solution**

$$\bar{\psi}(\vec{p}) = \langle\vec{p}|\psi\rangle = \iiint d\vec{r}\psi(\vec{r})\langle\vec{p}|\vec{r}\rangle = \iiint d\vec{r}\psi(\vec{r})\langle x|p_x\rangle\langle y|p_y\rangle\langle z|p_z\rangle$$
$$= \frac{1}{(2\pi\hbar)^{3/2}} \iiint d\vec{r} e^{-ixp_x/\hbar - iyp_y/\hbar - izp_z/\hbar}\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint d\vec{r} e^{-i\vec{p}\cdot\vec{r}/\hbar}\psi(\vec{r}).$$

(b) *We can define operators corresponding to $P_X, P_Y, P_Z$ in 3D as tensor products including 1D momentum operators: for example, $P_X := P_X \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z$. Show that they act on the wave position function in a following way:*

$$P_X \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z : \psi(\vec{r}) \to -i\hbar\frac{\partial}{\partial x}\psi(\vec{r})$$
$$\mathbb{I}_X \otimes P_Y \otimes \mathbb{I}_Z : \psi(\vec{r}) \to -i\hbar\frac{\partial}{\partial y}\psi(\vec{r})$$
$$\mathbb{I}_X \otimes \mathbb{I}_Y \otimes P_Z : \psi(\vec{r}) \to -i\hbar\frac{\partial}{\partial z}\psi(\vec{r})$$

**Solution** Let us show the action of the momentum operator $P_X$ (analogously for $P_Y$ and $P_Z$):

$$\langle\vec{r}|P_X \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z|\psi\rangle = \iiint d\vec{r'}\psi(\vec{r'})\langle\vec{r}|P_X \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z|\vec{r'}\rangle$$
$$= \iiint dx'dy'dz'\psi(x',y',z')\langle x|P_X|x'\rangle\langle y|y'\rangle\langle z|z'\rangle = \int dx'\psi(x',y',z')\langle x|P_X|x'\rangle$$
$$= -i\hbar\frac{\partial}{\partial x}\langle\vec{r}|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\psi(\vec{r})$$

(c) *Consider commutation relations among $X, Y, Z, P_X, P_Y, P_Z$. Show that all of the commutators are zero except for:*

$$[X, P_X] = [Y, P_Y] = [Z, P_Z] = i\hbar\mathbb{I}$$

**Solution** Let us look at the case of commutators with the $X$ operator. Trivially (by definition):

$$[X, X] = 0$$
$$[X, Y] = [X \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z, \mathbb{I}_X \otimes Y \otimes \mathbb{I}_Z] = 0 = [X, Z] = [X, P_Y] = [X, P_Z]$$

Now let us take a sample state $|\psi\rangle$:

$$\langle\vec{r}|[X,P_X]|\psi\rangle = \iiint dx'dy'dz'\langle x|[X,P_X]|x'\rangle\langle y|y'\rangle\langle z|z'\rangle\psi(x,y,z) = i\hbar\psi(\vec{r})$$
$$\Rightarrow [X,P_X] = i\hbar\mathbb{I}$$

(d) *Let us define $P^2 = P_X^2 + P_Y^2 + P_Z^2 := P_X^2 \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z + \mathbb{I}_X \otimes P_Y^2 \otimes \mathbb{I}_Z + \mathbb{I}_X \otimes \mathbb{I}_Y \otimes P_Z^2$. Show that $P^2$ acts on the wave position and momentum functions in following ways:*

$$P^2 : \psi(\vec{r}) \to -\hbar^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\psi(\vec{r})$$
$$P^2 : \bar{\psi}(\vec{p}) \to (P_X^2 + P_Y^2 + P_Z^2)\bar{\psi}(\vec{p})$$

**Solution** Given the relations we derived in (b), we arrive to:

$$\langle\vec{r}|P_X \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\psi(\vec{r}) \Rightarrow \langle\vec{r}|P_X^2 \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z|\psi\rangle = -\hbar^2\frac{\partial^2}{\partial x^2}\psi(\vec{r})$$

Also. for the momentum operator acting on the momentum wave function we have $\langle\vec{p}|P_X \otimes \mathbb{I}_Y \otimes \mathbb{I}_Z|\psi\rangle = p_X^2\langle\psi|(\vec{p})$. The relations in the exercise follow.

## Exercise 2. *Parity operator*

*Consider an operator $\mathcal{P}$ in 1D which represents the reflection about the point $x = 0$ (it is called the parity operator):*

$$\mathcal{P} : \psi(x) \to \psi(-x), \quad \mathcal{P}|x\rangle = |-x\rangle$$

(a) *Find the continuous matrix representation $\mathcal{P}(x,x')$ for the parity operator.*

**Solution** The continuous representation is given by:

$$P(x,x') = \langle x|\mathcal{P}|x'\rangle = \langle x|-x'\rangle = \delta(x+x')$$

(b) *Show that the only possible eigenvalues of $\mathcal{P}$ are $\pm 1$. What would it mean to measure $\mathcal{P}$ in the lab?*

**Solution** Let us note the following:

$$\langle x|\mathcal{P}^\dagger|x'\rangle = \langle -x|x'\rangle = \delta(x+x') \Rightarrow \mathcal{P}^\dagger = \mathcal{P};$$
$$\langle x|\mathcal{P}^\dagger\mathcal{P}|x'\rangle = \langle -x|-x'\rangle = \delta(x-x') \Rightarrow \mathcal{P}^\dagger\mathcal{P} = \mathbb{I}.$$

Hence, $\mathcal{P}$ is Hermitian *and* unitary, and the only possible eigenvalues are $\pm 1$.
If we measure the parity observable,

$$\langle\mathcal{P}\rangle = \langle\psi|\mathcal{P}|\psi\rangle = \iint \psi^*(x)\psi(x')\langle x|\mathcal{P}^\dagger|x'\rangle dxdx' = \iint \psi^*(x)\psi(x')\delta(x+x')dxdx' = \int \psi^*(x)\psi(-x)dx$$

If $\psi(x)$ is an even function, that is, $\psi(-x) = \psi(x)$, then $\langle\mathcal{P}\rangle = 1$; if $\psi(x)$ is an odd function, that is, $\psi(-x) = -\psi(x)$, then $\langle\mathcal{P}\rangle = -1$. Hence, the parity operator distinguishes between odd and even functions.

**Exercise 3.** ***Gaussian wave packets***

*Wave packets with various specific envelope functions $\phi(x)$ can be constructed. A common choice is the Gaussian wave packet, in which the envelope function has a Gaussian form:*

$$\psi(x) = \phi(x)e^{ikx} = \frac{1}{\sqrt[4]{2\pi\sigma^2}} \exp(-\frac{(x-x_0)^2}{4\sigma^2}) \cdot e^{ikx}$$

*(a) Determine the momentum wave function.*

**Solution** The momentum wave function is given by the Fourier transform of $\psi(x)$:

$$\bar{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ixp/\hbar} dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{1}{\sqrt[4]{2\pi\sigma^2}} \exp(-\frac{(x-x_0)^2}{4\sigma^2}) \cdot e^{-i(p-\hbar k)x/\hbar} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{1}{\sqrt[4]{4\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}) \cdot e^{-i(p-\hbar k)(x+x_0)/\hbar} dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{-i(p-\hbar k)x_0/\hbar} \int_{-\infty}^{\infty} \frac{1}{\sqrt[4]{2\pi\sigma^2}} \exp(-\frac{x^2}{4\sigma^2} - \frac{i(p-\hbar k)x}{\hbar}) dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}\sqrt[4]{2\pi\sigma^2}} e^{-i(p-\hbar k)x_0} \int_{-\infty}^{\infty} \exp\left( -\frac{1}{4\sigma^2}(x + \frac{2i\sigma(p-\hbar k)}{\hbar})^2 - \frac{\sigma^2(p-\hbar k)^2}{\hbar^2} \right) dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}\sqrt[4]{2\pi\sigma^2}} e^{-\frac{i(p-\hbar k)x_0}{\hbar} - \frac{\sigma^2(p-\hbar k)^2}{\hbar^2}} \sqrt{4\sigma^2\pi} = \sqrt[4]{\frac{2\sigma^2}{\pi\hbar^2}} \exp(-\frac{\sigma^2(p-\hbar k)^2}{\hbar^2}) \cdot e^{-i(p-\hbar k)x_0/\hbar}$$

The momentum wave function is also a Gaussian wave packet.

*(b) What are expected values of $x$ and $p$?*

**Solution**

$$\langle x \rangle = \langle \psi | X | \psi \rangle = \int_{-\infty}^{\infty} \psi^*(x) x \psi(x) dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x \exp(-\frac{(x-x_0)^2}{2\sigma^2}) dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \left( -\sigma^2 \frac{d}{dx} \exp(-\frac{(x-x_0)^2}{2\sigma^2}) + x_0 \exp(-\frac{(x-x_0)^2}{2\sigma^2}) \right) dx$$

$$= \frac{x_0}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp(-\frac{(x-x_0)^2}{2\sigma^2}) dx = x_0$$

The expected value of $x$ corresponds to the peak of the Gaussian.

$$\langle p \rangle = \langle \psi | P | \psi \rangle = \int_{-\infty}^{\infty} \psi^*(x)(-i\hbar \frac{d}{dx}) \psi(x) dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp(-\frac{(x-x_0)^2}{4\sigma^2}) e^{-ikx} (-i\hbar) \exp(-\frac{(x-x_0)^2}{4\sigma^2}) e^{ikx} \left( ik - \frac{x-x_0}{2\sigma^2} \right) dx$$

$$= \frac{-i\hbar \cdot ik}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp(-\frac{(x-x_0)^2}{2\sigma^2}) dx - \underbrace{\frac{-i\hbar}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp(-\frac{(x-x_0)^2}{2\sigma^2}) \frac{x-x_0}{2\sigma^2} dx}_{= 0 \text{ (integrated function is odd)}}$$

$$= \frac{\hbar k}{\sqrt{2\pi\sigma^2}} \sqrt{2\pi\sigma^2} = \hbar k$$

The expected value of $p$ is the speed of the movement of the packet.

*(c) Calculate variances $\langle \Delta x^2 \rangle$ and $\langle \Delta p^2 \rangle$.*

**Solution** The varience of the position: $\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$. Let us first calculate the expected value of $x^2$:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^*(x) x^2 \psi(x) dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 \exp(-\frac{(x-x_0)^2}{2\sigma^2}) dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x+x_0)^2 \exp(-\frac{x^2}{2\sigma^2}) dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \Bigg( \underbrace{\int_{-\infty}^{\infty} x^2 \exp(-\frac{x^2}{2\sigma^2}) dx}_{=\sqrt{2\pi\sigma^2}\sigma^2} + \underbrace{2x_0 \int_{-\infty}^{\infty} x \exp(-\frac{x^2}{2\sigma^2}) dx}_{=0} + \underbrace{x_0^2 \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2\sigma^2}) dx}_{\sqrt{2\pi\sigma^2} x_0^2} \Bigg) = x_0^2 + \sigma^2$$

It follows that $\langle \Delta x^2 \rangle = \sigma^2$.

For the momentum we have:

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \bar{\psi}^*(p) p^2 \bar{\psi}(p) dx = \sqrt{2\frac{\sigma^2}{\pi\hbar^2}} \int_{-\infty}^{\infty} p^2 \exp(-\frac{2\sigma^2(p-\hbar k)^2}{\hbar^2}) dp$$

$$= \sqrt{\frac{2\sigma^2}{\pi\hbar^2}} \Bigg( \underbrace{\int_{-\infty}^{\infty} p^2 \exp(-\frac{2p^2\sigma^2}{\hbar^2}) dp}_{=\sqrt{\pi/2}\hbar^3/(4\sigma^3)} + \underbrace{2\hbar k \int_{-\infty}^{\infty} p \exp(-\frac{2p^2\sigma^2}{\hbar^2}) dp}_{=0} + \underbrace{\hbar^2 k^2 \int_{-\infty}^{\infty} \exp(-\frac{2p^2\sigma^2}{\hbar^2}) dp}_{\sqrt{\pi/2}\hbar^3 k^2/\sigma} \Bigg)$$

$$= \hbar^2 k^2 + \frac{\hbar^2}{4\sigma^2}$$

It follows that $\langle \Delta p^2 \rangle = \frac{\hbar^2}{4\sigma^2}$.

(d) *For which values of $x_0$ and $\sigma$ is the Gaussian wave packet an eigenstate of the parity operator $\mathcal{P}$? What is the associated eigenvalue?*

**Solution** We have established in the previous exercise that the eigenfuctions of the parity operator are even (eigenvalue +1) and odd (eigenvalue -1) functions. Hence, to be an eigenfunction of the parity operator, the Gaussian wave packet has to have $x_0 = 0$ – in this case it constitutes an even function (for any $\sigma$). The associated eigenvalue is then +1.

## Exercise 4. *Entanglement and teleportation*

*Imagine that Alice (A) has a pure state $|\phi\rangle_S$ of a system $S$ in her lab. She wants to send that state to Bob, who lives, of course, on the Moon, but she does not trust the postwoman Eve to carry it there personally. Here, we will see that if Alice and Bob share an entangled state Alice can "teleport" the state $|\phi\rangle$ to the system $B$ that Bob controls.*

*Formally, we have three systems $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. In this exercise we will assume all three are qubits. The initial state is*

$$|\phi\rangle_S \otimes \frac{1}{\sqrt{2}} \left(|00\rangle_{AB} + |11\rangle_{AB}\right), \tag{1}$$

*i.e. $S$ is decoupled from $A$ and $B$ and these two are fully entangled in a Bell state (they can create this state by starting in states $|0\rangle_A$ and $|0\rangle_B$ and applying Hadamard and CNOT gates in sequence, as shown on the Figure 1). We may write $|\phi\rangle_S = \alpha|0\rangle_S + \beta|1\rangle_S$.*

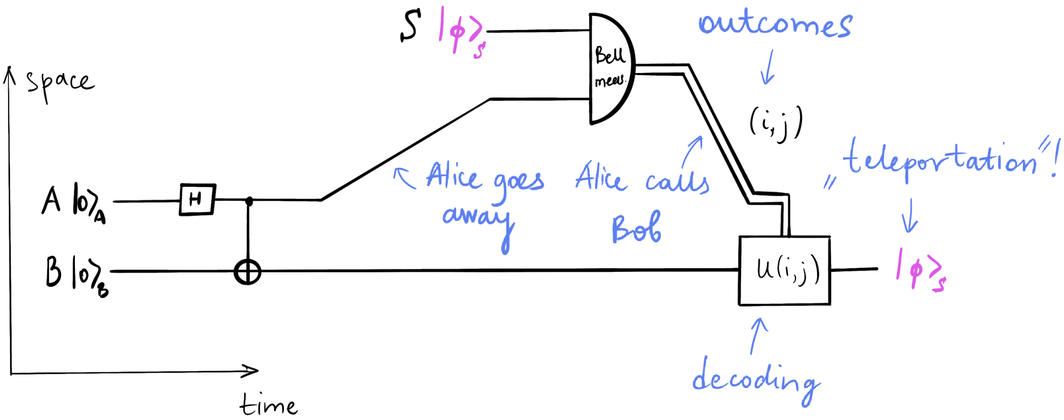


Figure 1: The quantum circuit for the quantum state teleportation.

*(a) In a first step, Alice will measure systems $S$ and $A$ jointly in the Bell basis, that is, the projectors that represent different outcomes are $\{|\psi_i\rangle\langle\psi_i|_{SA}\}_{i=1,..,4}$, where*

$$\tfrac{1}{\sqrt{2}}\left(|00\rangle_{SA}+|11\rangle_{SA}\right),\ \tfrac{1}{\sqrt{2}}\left(|00\rangle_{SA}-|11\rangle_{SA}\right),\ \tfrac{1}{\sqrt{2}}\left(|01\rangle_{SA}+|10\rangle_{SA}\right),\ \tfrac{1}{\sqrt{2}}\left(|01\rangle_{SA}-|10\rangle_{SA}\right).$$

*Then Alice communicates (classicaly) the result of her measurement to Bob. What is the reduced state of Bob's system ($B$) for each of the possible outcomes?*

**Solution** The joint state of the systems $S$, $A$ and $B$ just before Alice's measurement:

$$|\psi\rangle_{SAB}=\frac{\alpha}{\sqrt{2}}|000\rangle_{SAB}+\frac{\alpha}{\sqrt{2}}|011\rangle_{SAB}+\frac{\beta}{\sqrt{2}}|100\rangle_{SAB}+\frac{\beta}{\sqrt{2}}|111\rangle_{SAB}$$

Now let us analyse four possible cases of outcomes that Alice can get after measuring systems $S$ and $A$ jointly:

1. **Alice measures** $\frac{1}{\sqrt{2}}\left(|00\rangle_{SA}+|11\rangle_{SA}\right)$. In this case, the reduced state of $B$ after the measurement is (up to normalization)

$$|\chi\rangle_B=\tfrac{1}{\sqrt{2}}\left(\langle 00|_{SA}+\langle 11|_{SA}\right)|\psi\rangle_{SAB}$$
$$=\tfrac{1}{\sqrt{2}}\left(\langle 00|_{SA}+\langle 11|_{SA}\right)\left(\frac{\alpha}{\sqrt{2}}|000\rangle_{SAB}+\frac{\alpha}{\sqrt{2}}|011\rangle_{SAB}+\frac{\beta}{\sqrt{2}}|100\rangle_{SAB}+\frac{\beta}{\sqrt{2}}|111\rangle_{SAB}\right)$$
$$=\frac{\alpha}{2}|0\rangle_B+\frac{\beta}{2}|1\rangle_B$$

   After normalization, we arrive to $|\chi\rangle_B=\alpha|0\rangle_B+\beta|1\rangle_B$.

2. **Alice measures** $\frac{1}{\sqrt{2}}\left(|00\rangle_{SA}-|11\rangle_{SA}\right)$. In this case, the reduced state of $B$ after the measurement is (up to normalization)

$$|\chi\rangle_B=\tfrac{1}{\sqrt{2}}\left(\langle 00|_{SA}-\langle 11|_{SA}\right)|\psi\rangle_{SAB}$$
$$=\tfrac{1}{\sqrt{2}}\left(\langle 00|_{SA}-\langle 11|_{SA}\right)\left(\frac{\alpha}{\sqrt{2}}|000\rangle_{SAB}+\frac{\alpha}{\sqrt{2}}|011\rangle_{SAB}+\frac{\beta}{\sqrt{2}}|100\rangle_{SAB}+\frac{\beta}{\sqrt{2}}|111\rangle_{SAB}\right)$$
$$=\frac{\alpha}{2}|0\rangle_B-\frac{\beta}{2}|1\rangle_B$$

   After normalization, we arrive to $|\chi\rangle_B=\alpha|0\rangle_B-\beta|1\rangle_B$.

3. **Alice measures** $\frac{1}{\sqrt{2}}(|01\rangle_{SA} + |10\rangle_{SA})$. In this case, the reduced state of $B$ after the measurement is (up to normalization)

$$|\chi\rangle_B = \tfrac{1}{\sqrt{2}}(\langle 01|_{SA} + \langle 10|_{SA})\,|\psi\rangle_{SAB}$$
$$= \tfrac{1}{\sqrt{2}}(\langle 01|_{SA} + \langle 10|_{SA})\,(\frac{\alpha}{\sqrt{2}}|000\rangle_{SAB} + \frac{\alpha}{\sqrt{2}}|011\rangle_{SAB} + \frac{\beta}{\sqrt{2}}|100\rangle_{SAB} + \frac{\beta}{\sqrt{2}}|111\rangle_{SAB})$$
$$= \frac{\alpha}{2}|1\rangle_B + \frac{\beta}{2}|0\rangle_B$$

After normalization, we arrive to $|\chi\rangle_B = \alpha|1\rangle_B + \beta|0\rangle_B$.

4. **Alice measures** $\frac{1}{\sqrt{2}}(|01\rangle_{SA} - |10\rangle_{SA})$. In this case, the reduced state of $B$ after the measurement is (up to normalization)

$$|\chi\rangle_B = \tfrac{1}{\sqrt{2}}(\langle 01|_{SA} - \langle 10|_{SA})\,|\psi\rangle_{SAB}$$
$$= \tfrac{1}{\sqrt{2}}(\langle 01|_{SA} - \langle 10|_{SA})\,(\frac{\alpha}{\sqrt{2}}|000\rangle_{SAB} + \frac{\alpha}{\sqrt{2}}|011\rangle_{SAB} + \frac{\beta}{\sqrt{2}}|100\rangle_{SAB} + \frac{\beta}{\sqrt{2}}|111\rangle_{SAB})$$
$$= \frac{\alpha}{2}|1\rangle_B - \frac{\beta}{2}|0\rangle_B$$

After normalization, we arrive to $|\chi\rangle_B = \alpha|1\rangle_B - \beta|0\rangle_B$.

(b) *Depending on the classical outcomes of the measurement by Alice, Bob needs to apply a combination of two operators from the set* $\{X, Z, \mathbb{I}\}$ *(where* $X$*,* $Z$ *are Pauli operators). What is this combination?*

**Solution** Coming back to the cases discussed above, we can come up with a following strategy:

1. **Alice measures** $\frac{1}{\sqrt{2}}(|00\rangle_{SA} + |11\rangle_{SA})$. The reduced state of B is $|\chi\rangle_B = \alpha|0\rangle_B + \beta|1\rangle_B$ – Bob doesn't need to apply any operations to restore $|\phi\rangle_B$ on his side.
2. **Alice measures** $\frac{1}{\sqrt{2}}(|00\rangle_{SA} - |11\rangle_{SA})$. The reduced state of B is $|\chi\rangle_B = \alpha|0\rangle_B - \beta|1\rangle_B$ – Bob needs to apply a $Z$ gate to restore $|\phi\rangle_B$ on his side ($Z(\alpha|0\rangle_B - \beta|1\rangle_B) = \alpha|0\rangle_B + \beta|1\rangle_B$).
3. **Alice measures** $\frac{1}{\sqrt{2}}(|01\rangle_{SA} + |10\rangle_{SA})$. The reduced state of B is $|\chi\rangle_B = \alpha|1\rangle_B + \beta|0\rangle_B$ – Bob needs to apply an $X$ gate to restore $|\phi\rangle_B$ on his side ($X(\alpha|1\rangle_B + \beta|0\rangle_B) = \alpha|0\rangle_B + \beta|1\rangle_B$).
4. **Alice measures** $\frac{1}{\sqrt{2}}(|01\rangle_{SA} - |10\rangle_{SA})$. The reduced state of B is $|\chi\rangle_B = \alpha|1\rangle_B - \beta|0\rangle_B$ – Bob needs to apply a combination of $Z$ and $X$ gates to restore $|\phi\rangle_B$ on his side ($(ZX(\alpha|1\rangle_B - \beta|0\rangle_B) = \alpha|0\rangle_B + \beta|1\rangle_B)$).

Thus, we have successfully "teleported" a quantum state from Alice's to Bob's lab.